Parsimonious exposure-at-default modeling for unfunded loan commitments

Pinaki Bag
Union National Bank, Abu Dhabi, UAE, and
Michael Jacobs Jr
Credit Risk Analysis Division, US Office of the Comptroller of the Currency, Washington, DC, USA

Abstract

Purpose – The purpose of this paper is to build an easy to implement, pragmatic and parsimonious yet accurate model to determine an exposure at default (EAD) distribution for CCL (contingent credit lines) portfolios.

Design/methodology/approach – Using an algorithm similar to the basic CreditRisk + and Fourier Transforms, the authors arrive at a portfolio level probability distribution of usage.

Findings – The authors perform a simulation experiment which illustrates the convolution of two portfolio segments to derive an EAD distribution, chosen randomly from Moody's Default Risk Service (DRS) database of CCLs rated as of 12/31/2008, to derive an EAD distribution. The standard deviation of the usage distribution is found to decrease as we increase the number of puts used, but the mean value remains relatively stable, as the extreme points converge towards the mean to produce a shrinkage in the spread of the distribution. The authors also observe, for the sample portfolio, that an increase in the additional usage rate level also increases the volatility of the associated exposure distribution.

Practical implications – This model, in conjunction with internal bank financial institution research, can be used for banks' EAD estimation as mandated by Basel II for bank CCL portfolios, or implemented as part of a Solvency II process for insurers exposed to credit sensitive unfunded commitments. Apart from regulatory requirements, distributions of stochastic exposure generated can be inputs for different economic capital models and stress testing procedures used to capture an accurate risk profile of the portfolio, as well as providing better insights into the problem of managing liquidity risk for a portfolio of CCLs and similar exposures.

Originality/value – In-spite of the large volume of CCLs in portfolios of financial institutions all (for commercial banks holding these as well as for insurance companies having analogous exposures), paucity of EAD models, unsuitability of external data and inconsistent internal data with partial draw-downs have been a major challenge for risk managers as well as regulators in managing CCL portfolios.

Keywords Portfolio investment, Risk management, Probability theory, Exposure-at-default, Basel II, Solvency II, Credit risk, Contingent credit

Paper type Technical paper

JEL classification – G20, G21, C13

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I. Introduction

The Bank for International Settlement in its Basel II guidelines[1] describes capital as a function of the probability of default (PD), loss-given-default (LGD) and exposure-at-default (EAD), with all playing an equally vital role; but a simple Google search on each term returns 5.7 million, 65,000 and 29,000[2] hits, respectively. While a Google search may not be concrete enough to draw firm conclusions, it does indicate the trends in terms of importance given to each component by academicians and practitioners. Similar assertions are made by the Financial Services Authority (FSA)[3], UK regarding EAD models as well. It is worth noting in this regard that while regulatory capital (and in many models, economic capital as well) is generally concave in PD, it is linear or even convex in EAD (as well as LGD), depending upon the factor structure. EAD modeling is also an important concern for insurance companies, which must as part of their Solvency II processes estimate both credit loss due to counterparty defaults as well as risk due to unexpected claims, with many of such exposures have structural similarities to contingent credit lines (CCLs) (Eling *et al.*, 2007)[4]. This highlights the importance of the EAD parameter in the face of a shortage of modeling efforts targeting such.

CCL or loan commitments[5] are contractual promises by a financial institution to specific obligors to lend up to specified limits on pre-determined rates and terms. In general, these are accompanied by different fees, to be paid over the life of the commitment. Furthermore, there is often a material adverse change clause, which states that the firm may cancel the line if the credit quality of the specific obligor deteriorates.

According to a Federal Deposit Insurance Corporation survey[6] close to 80 percent of all commercial and industrial loans are structured as CCL contracts, and as of September 2009, the unused CCLs of US firms were close to $1.9 trillion. Avery and Berger (1991) state that main reason for using credit commitments is to provide flexibility during a slowdown, and as noted by Kanatas (1987) these can be seen as hedging instruments. Hawkins (1982) comments that credit lines help borrowers manage fluctuations in working capital, but Sufi (2009) reports that firms with low cash flow or high cash flow volatility rely more heavily on cash rather than credit lines.

Basel II guidelines calculate a regulatory capital charge on contingent credit commitments based on a credit conversion factor (CCF) and a risk weight, either of which lying between 0 and 100 percent. As noted by Hull (1989) for Basel I, CCF for a small bank underestimates the capital requirement, as the "fat tails" effect increases the capital requirement proportionately more for off-balance sheet items. In the Basel II advanced internal rating-based (AIRB) methodology for computing regulatory capital, banks are allowed to compute their own estimates of the CCF, and therefore its own estimate of EAD for CCL, provided those estimates can be supported empirically by data relevant to the banks' current portfolio. In the case of insurance, there is a an analogous construct for reinsurance default risks, in which firms have the option of pursuing alternative risk sensitive frameworks to quantify uncertain potential exposures to the default of other insurers[7]. Apart from the lack of EAD models constructed by consultants and academics, leading to a paucity of external data and models, other issues identified by FSA include the scarcity of usable data on draw-downs for defaulted CCLs in each institution and unsuitability of external data. In cases where external data are available, the relevance of this is often open to suspicion, as the estimates are strongly influenced by lender's behavior.

The current paper is an attempt to build an easy to implement, parsimonious yet accurate model to estimate potential exposure for a CCL portfolio. Each CCL is modeled as a portfolio of options written to the obligors, which they can exercise with the firm at pre-specified terms and conditions. Modeling the exercise of options as a Poisson process, a distribution of stochastic exposure for different segments at a portfolio level is constructed. We use a standard Fast Fourier Transform (FFT) algorithm in order to convolute these portfolio segments and generate an exposure probability distribution of the complete portfolio.

This model, in conjunction with internal bank research, can be used for banks' EAD estimation as mandated by Basel II for CCL portfolios, or by insurance companies to quantify the component of counterparty credit risk with respect to unfunded commitments as mandated by Solvency II. Apart from regulatory requirements, distributions of stochastic exposure generated can be inputs for different economic capital models and stress testing procedures used to capture an accurate risk profile of the portfolio. This will also contribute in providing better insights into the problem of managing liquidity risk for a portfolio of CCLs in the case of banks (e.g. credit cards or home equity line of credits (HELC)), or for analogous exposures to insurers (e.g. reinsurance contracts).

This paper will proceed as follows. In Section II we will review past studies related to CCLs. In Section III we detail how we can use options executions to link the draw-downs in a portfolio to the distribution of potential CCL exposure. Section IV provides an implementation using a hypothetical portfolio. Finally, in Section V we conclude with implications of this work and potential avenues for future research.

II. CCL, options and partial draw-down

Estimation of EAD for CCLs has followed different algorithms, giving rise to different mathematical representations. In the first, an appropriate conversion factor (CF) is formed by multiplication with the total limit (TL) of the facility:

$$\mathrm{EAD} = \mathrm{CF} \cdot \mathrm{TL} \tag{1}$$

In the second, we form an appropriate conversion factor (α)[8] that is multiplied with the unused part of the limit (L) of the facility:

$$\mathrm{EAD} = \mathrm{Current\ \ Exposure} + \alpha \cdot L \tag{2}$$

In a third form, a suitable conversion factor is multiplied with the current exposure:

$$\mathrm{EAD} = \mathrm{CF} \cdot \mathrm{Current\ \ Exposure} \tag{3}$$

Jacobs (2010) refers to the factor CF in equation (3) as the CCF. As shown by Moral (2006), both equations (1) and (2) will yield same results for EAD estimation except when the CCL fully utilized[9]. As discussed by Miu and Ozdemir (2008), the Basel II CCF is generally equivalent to α in equation (2). We will use equation (2) for all our subsequent analysis, as it is most prevalent in practice for this purpose. As current exposure and L are known, modeling α (i.e. the partial draw-down of the unused limit), we believe will give us clearer insight into the problem of EAD estimation for CCLs.

Studies related to CCLs have focused either on pricing of these instruments[10] or on the level of partial draw-down in each credit line. In this study, we will concentrate on latter through the development of a model having risk management applications, including the generation of a more accurate credit loss distribution.

Thakor *et al.* (1981) employ an option-theoretic[11] approach to pricing CCLs, modeling them as puts written by the bank, and measure the sensitivity of the values of such to changes in interest rates. Partial draw-downs on a CCL are explained through interest elasticity of demand for borrowed funds and bank customer relationship dynamics. If a firm has infinite opportunities for investment and no restriction on capital structure or leverage, then the interest elasticity will be perfectly elastic, and vice-versa if there are no such opportunities, in which case it will be perfectly inelastic. Alternatively, if we look into the bank-customer relationship framework, a customer will try to minimize its expected cost of renewal on the line in next year and opportunity for loss by not utilizing the full facility this year when availing the line.

Kaplan and Zingales (1997) find that the un-drawn portion of credit lines decreases when firms are more liquidity constrained. Gatev and Strahan (2006)[12] report that partial draw-downs increase when the commercial paper – T bill rate spread rises. Both of these studies indicate presence of a pricing incentive for partial draw-downs, induced by interest rate fluctuations, which subsequently inspired Jones and Wu (2009) to use the same for modeling framework for partial draw-downs. They model credit quality as a jump-diffusion process, giving rise to partial draw-downs and CCL pricing as a function of the dynamic credit state. The proportion of the credit line drawn is modeled as function of the difference between an alternative opportunity rate and the marginal cost of line borrowings. This opportunity rate is defined as the rate of interest charged to the borrower if she borrows outside the purview of the defined credit line. In order to incorporate linking of the loan spread to the credit default swap spread of the borrower, the marginal cost of borrowing is defined as function of the reference rate, contractual spread over the reference rate and proportion of the excess that is added to current period loan. Apart from the interest rate differential, the sensitivity of drawdown to the interest rate differential is included to model the amount of partial draw-down. This sensitivity is similar to the interest elasticity proposed by Thakor *et al.* (1981).

Both of these approaches (Thakor *et al.*, 1981; Jones and Wu, 2009) appear intuitive and convincing, but implementing such in banks where most of the CCL are extended to unrated obligors whose market spread may not be easily available might pose problems of parameterization. Some parameters, such as interest elasticity, may be affected by firm-specific behavior as well as present macroeconomic factors. Another approach to estimating usage of limits is in a continuous-time model, where the credit provider and the credit taker interact within a game-theoretic framework, as has been attempted by Leippold *et al.* (2003).

Attempts to directly estimate partial draw-down has also been undertaken in previous empirical literature. Asarnow and Marker (1995) present partial draw-down estimates based on credit lines issued by Citibank to publicly rated North American firms over the five-year period from 1987 to 1992. They find a downward sloping pattern of usage level from high to low rated obligors (i.e. lower rated firms would have already consumed their credit lines earlier than when it approaches default). A similar trend is also noted by Araten and Jacobs (2001), where partial draw-downs decrease as the firm approaches default. Their estimate of the partial draw-down rate (the "loan equivalency factor (LEQ)") is based on 1,021 observations (408 facilities of 399 borrowers) at a quarterly frequency in the period 1Q 1995-4Q 2000 for J.P. Morgan chase borrowers. They also report that level of usage is influenced by risk rating but not by commitment size or borrower industry. Jacobs' (2010) empirical study is based on a dataset encompassing 544 defaulted instruments from 496 US borrowers with public credit ratings over the period from 1985 to 2007. He finds similar trends in terms of additional drawdown and further points out the statistically significant effect of obligor profits on the level of usage. He also notes that EAD risk (i.e. the LEQ factor or additional drawdown rate) is generally lower during downturns.

Agarwal and Ambrose (2006) examine utilization of HELC in the US market and confirm that borrowers with deteriorating credit quality increase their utilization. Jiménez *et al.* (2008) document credit line usage of different firms granted by banks in Spain between 1984 and 2005. The final dataset consists of 696,445 credit lines granted to 334,442 firms by 404 banks. They report a variety of factors such as commitment size, collateralization and maturity of CCLs that affect the usage level. They also report a statistically significant higher usage rate for firms that eventually default at least three years prior to default and that the usage monotonically increases as these firms approach default.

It is argued in several studies (Asarnow and Marker, 1995; Araten and Jacobs, 2001) that the level of usage is mainly affected by two distinct forces, that the lender might detect deteriorating credit quality of the borrower and cut back the limits (thereby increasing the utilization ratio), or that the borrower may actually use up the line before the lender realizes the deteriorating credit quality. As indicated by Qi (2009), examining credit card usage in the USA, borrowers are more active than lenders in this game of "race to default." Martin and Santomero (1997) analyze the pricing of CCL from the demand side of firms and show that credit line usage depends on the business growth potential of the firm as well as the uncertainty involved in those investment opportunities. External macroeconomic variables – such as size of credit line, collateralization, etc. – are also found to be associated with the level of partial draw-downs of obligors on CCLs.

We close this section by reviewing a relevant study from the parallel insurance risk modeling literature, which addresses similar methodological issues from the point of view of insurers subject to uncertain exposure conditional on counterparty default. Britt and Krvavych (2001) present the stochastic dynamic financial analysis approach to reinsurance credit risk, which models reinsurer default as the outcome of an asset impairment event being dependent on the current state of global reinsurance market. The state-conditional default rates based on reinsurance credit rating are calibrated by using industry available impairment rates for reinsurers and matching the model implied default dependency structure to the target one of the underlying assets. The proposed model is capable of overcoming common shortcoming arising from applying the existing investment banking concepts to calibrating credit risk in reinsurance sector, including that it allows for stochastic EAD quantities.

III. Partial draw-down at portfolio level

In order to put ourselves on firmer footing, let us define a bank having a CCL portfolio of N obligors each having one facility of unused CCL, or an insurer having exposures to a set of N reinsurance contracts where the severity of the uncompensated loss is uncertain (for brevity, we will denote either of these exposures as a "CCL"). Each of these exposures of the institution can arise before the expiry of the contract (either a credit line used or a default on an insured event), which herein – for the sake of tractability – we assume to be constant. For obligor A with CCL size of $L_{A,}$ we can safely assume that A has a very large number of put options which she can choose to exercise, and that the number of instruments she will exercise will determine the level of partial draw-down (or of defaulted reinsurance obligation). If she exercises all the puts, then she will have consumed her whole limit. We assume that she has n puts at her disposal, where n is sufficiently large. Furthermore, the size of each put (i.e. the strike price in this case) can be given as:

$$Q_A = \frac{L_A}{n} \tag{4}$$

Therefore, the amount of partial draw-down can be given as $r \times Q_A$ where r is the number of puts exercised by A in the period under consideration. It follows that we may define that the probability generating function (PGF) of r, the random number of options exercise, as:

$$F_A(Z) = \mathrm{P}(\mathrm{r}=0)Z^0 + \mathrm{P}(\mathrm{r}=1)Z^1 + \mathrm{P}(\mathrm{r}=2)Z^2 + \mathrm{P}(\mathrm{r}=3)Z^3 + \mathrm{P}(\mathrm{r}=4)Z^4 \ldots \tag{5}$$

Let us assume that the expected usage of the CCL is $\alpha_{A,}$so the average number of puts used by A is:

$$\lambda_A = \frac{\alpha_A L_A}{Q_A} \tag{6}$$

Using a Poisson process of exercise for each option we have that the PGF given by equation (5) reduces to:

$$F_A Z = \frac{e^{-\lambda_A}\lambda_A^0}{0!}Z^0 + \frac{e^{-\lambda_A}\lambda_A^1}{1!}Z^1 + \frac{e^{-\lambda_A}\lambda_A^2}{2!}Z^2 + \cdots = e^{-\lambda_A}\cdot e^{\lambda_A Z} \tag{7}$$

For all obligors, m ($m \leq N$) in the portfolio having put size equal to Q′, we can have the PGF for r number of puts being used as follows, which holds assuming independence of obligors in exercising of each option:

$$\prod_{\mathrm{A}=1}^{m} F_A z = \prod_{\mathrm{A}=1}^{m} e^{-\lambda_A} e^{\lambda_A z} = e^{-\sum_{\mathrm{A}=1}^{m}\lambda_A + \sum_{\mathrm{A}=1}^{m}\lambda_A z} \tag{8}$$

Now let us assume that the overall expected additional usage on the unused in the portfolio is α and the unused limits in this portfolio of m the obligors to be $L_{A,}$which implies that:

$$\alpha = \frac{\sum_{\mathrm{A}=1}^{m}\alpha_A L_A}{\sum_{\mathrm{A}=1}^{m} L_A} \tag{9}$$

Let $S = \sum_{\mathrm{A}=1}^{m}\lambda_A$, and as we have assumed all the size of every put is constant at Q, we can replace Q_A by Q in this sub-portfolio and let:

$$\beta = \mathrm{S}\cdot Q \tag{10}$$

Therefore:

$$\mathrm{S} = \sum_{\mathrm{A}=1}^{m}\lambda_A = \sum_{\mathrm{A}=1}^{m}\frac{\alpha_A\cdot L_A}{Q_A} = \frac{\sum_{\mathrm{A}=1}^{m}\alpha_A L_A}{Q} = \frac{\alpha}{Q}\cdot\sum_{\mathrm{A}=1}^{m} L_A \tag{11}$$

Therefore, with the PGF of usage being equal to r X Q for this sub-portfolio and put size Q, combining equations (8) and (11) yields:

$$\prod_{A=1}^{m} F_A Z = \mathrm{F_Q} Z = e^{Z-1S} = e^{-S} \cdot \left[\frac{S \cdot Z^0}{0!} + \frac{S \cdot Z^1}{1!} + \frac{S \cdot Z^2}{2!} + \cdots\right] \tag{12}$$

Now for any real portfolio, Q will not be equal for every obligor. Simplifying our calculation, we segregate the portfolio into different sub-portfolios, such that obligors in each sub-portfolio will have same put size of Q_i. In order to be conservative in our approach we will round Q up to the next unit[13]. Since in each sub-portfolio Q_i is the same whenever there is exercise of a put, then there is usage of $1XQ_i$. Furthermore, if there is exercise of two puts there is usage of $2XQ_i$ so we can write:

$$Pr(Usage = r \cdot Q_i) = \Pr(r \;\; puts \;\; being \;\; used) \tag{13}$$

Therefore, we can write the PGF of the sub-portfolio usage as:

$$F_{Q_i} Z = \sum_{\mathrm{r}=0}^{\infty} P(\mathrm{Usage} = \mathrm{r} \cdot Q_i) Z^{r \cdot Q_i} = \sum_{\mathrm{r}=0}^{\infty} P(\mathrm{Usage} = \mathrm{r}) Z^{r \cdot Q_i} \tag{14}$$

$$F_{Q_i} Z = \sum_{\mathrm{r}=0}^{\infty} e^{-S_i} \frac{S_i^r}{\mathrm{r}!} Z^{r \cdot Q_i} = e^{-S_i} e^{S_i Z^{Q_i}} \tag{15}$$

Hence, in order to find the usage distribution of the whole portfolio, we must first convolute each of these sub-portfolios. Hence, for the whole portfolio with t sub-portfolios, we can write the PGF as (assuming the independence of each sub-portfolio):

$$F_{Q'} Z = \prod_{\mathrm{i}=1}^{t} F_{Q_i}(z) = \prod_{\mathrm{i}=1}^{t} e^{-S_i} e^{S_i Z^{Q_i}} = e^{-\sum_{\mathrm{i}=1}^{t} s_i + \sum_{\mathrm{i}=1}^{t} S_i Z^{Q_i}} \tag{16}$$

In order to determine the exposure distribution of the portfolio, it follows from Taylor's theorem that:

$$P(\mathrm{Usage} = \mathrm{r} \cdot Q_i) = \frac{1}{\mathrm{r}!} \left. \frac{d^r (F_{Q'}(Z))}{\mathrm{d}z^r} \right|_{\mathrm{Z}=0} \quad \text{for } \; \mathrm{r} = 0, 1, 2, \ldots \tag{17}$$

Let:

$$W_r = \frac{1}{\mathrm{r}!} \left. \frac{d^r F_{Q'}(Z)}{\mathrm{d}z^r} \right|_{Z=0} = \frac{1}{\mathrm{r}!} \frac{d^{r-1}}{\mathrm{d}z^{r-1}} \left. \frac{d(F_{Q'}(Z))}{\mathrm{d}z} \right|_{Z=0} \tag{18}$$

Since $\sum_{\mathrm{i}=1}^{t} S_i$ is constant:

$$W_r = \frac{1}{\mathrm{r}!} \frac{d^{r-1}}{\mathrm{d}z^{r-1}} \left. \left(F_{Q'}(Z) \cdot \frac{d\left(\sum_{\mathrm{i}=1}^{t} S_i Z^{Q_i}\right)}{\mathrm{d}z} \right) \right|_{\mathrm{z}=0} \tag{19}$$

By Leibnitz's formula for *n*th order differentiation we have:

$$W_r = \frac{1}{\mathrm{r}!}\sum_{\mathrm{k}=0}^{r-1}\binom{r-1}{k}\cdot\frac{d^{r-1-k}F_{Q'}(Z)}{\mathrm{d}z^{r-1-k}}\cdot\frac{d^{\mathrm{k}+1}\left(\sum_{\mathrm{i}=1}^{t}S_iZ^{Q_i}\right)}{\mathrm{d}z^{\mathrm{k}+1}}\Bigg|_{\mathrm{Z}=0} \tag{20}$$

If $Q_i = \mathrm{K}+1$ and $\mathrm{Z}=0$, then:

$$\frac{d^{\mathrm{k}+1}\left(\sum_{\mathrm{i}=1}^{t}S_iZ^{Q_i}\right)}{\mathrm{d}z^{\mathrm{k}+1}} = (\mathrm{k}+1)!\mathrm{S}_i \tag{21}$$

Else, if $Q_i \neq \mathrm{K}+1$ we have:

$$\frac{d^{\mathrm{k}+1}\left(\sum_{\mathrm{i}=1}^{t}S_iZ^{Q_i}\right)}{\mathrm{d}z^{\mathrm{k}+1}} = 0 \tag{22}$$

In addition, we have from equation (18):

$$W_{r-1-k} = \frac{1}{(r-1-k)!}\frac{d^{r-1-k}F_{Q'}(Z)}{\mathrm{d}Z^{r-1-k}}\Bigg|_{\mathrm{Z}=0} \tag{23}$$

Hence combining equations (20)-(23) at $\mathrm{Z}=0$ we have:

$$\begin{aligned} W_r &= \frac{1}{\mathrm{r}!}\sum_{\mathrm{k}=\mathrm{Q}_i-1}^{r-1} r-1 \quad C_k\cdot(\mathrm{k}+1)!\mathrm{S}_i\cdot(r-1-k)!\mathrm{W}_{r-1-k}\big|_{\mathrm{Z}=0} \\ &= \frac{1}{r}\sum_{\mathrm{i};\mathrm{Q}_i\leq r} Q_i\cdot S_i\cdot W_{r-Q_i} \end{aligned} \tag{24}$$

Hence from equations (24) and (10):

$$W_r = \frac{1}{r}\sum_{\mathrm{i};\mathrm{Q}_i\leq r}\beta_i\cdot W_{r-Q_i} \tag{25}$$

Now if we have $\mathrm{Q_i}$ with integers starting from one to a large enough integer we can get the complete probability distribution of portfolio usage, starting from:

$$W_0 = \mathrm{e}^{-\sum_{\mathrm{i}=1}^{t}S_i}$$

If we set $\mathrm{Q_i}$ to 1, we will have exposure distribution from \$1 (as seen from equation (3)), where $\mathrm{Q_i}$ represents the size of each put.

Thus, far we have used a constant α for the whole portfolio, but in reality we will have a situation where α will be different for each segment of the portfolio depending upon product type, risk rating and other factors. As noted in a survey[14], banks generally prefer to use segment-wise α for its EAD estimation. Hence after performing the computations leading to equation (25), we will be left with probability distributions of usage for different segments of the portfolio. These segments can be at different loan types or facility ratings. In order to arrive finally at a portfolio-level exposure distribution we will follow a standard convolution procedure using Fourier Transforms.

For simplicity, let us assume that we have only two distinct segment of the CCL portfolio. Then from equation (25) we can have two distinct vectors ($F = f_0, f_1, \ldots, f_{1-1}$ and $G = g_0, g_1, \ldots, g_{m-1}$) representing the probabilities of usage for each segment. Let R represent the vector formed by convolution of F and G. In order to perform FFT[15], we will pad each of the vector such that length of each vector is s, where $s \geq 1 + m$ and s are of the form 2^x where x is an integer. We know from the convolution theorem that:

$$R = F \otimes G = \text{Inverse FFT}(\text{FFT}(F) \cdot \text{FFT}(G)) \quad (26)$$

Hence R will give us the overall portfolio usage probability distribution. This procedure can easily be replicated if we have more than two segments in our portfolio.

IV. Numerical experiment

For a typical CCL portfolio, ΣS_i for the whole portfolio may be quite large, and we are trying to assign probability to each dollar of usage in the algorithm, so it may be daunting task to derive the full distribution. In the calculation of the negative exponential of a very large number (ΣS_i), for initiating the calculation (as $W_{0,}$) we are confronted with precision issues. That is, the double-precision Monniaux (2008) regime of most common software applications (including Matlab, Octave or Microsoft Excel) under default settings will approximate W_0 as zero, and as the derivation of the full distribution depends on W_0, such distribution would be evaluated incorrectly.

There are potentially many alternatives to circumvent the problem in standard applications. One solution to this problem may be use of libraries, which can handle very high precision calculations[16]. This may also require higher computational power in terms of hardware, the specification of which is beyond the scope of the current paper.

In order to illustrate these calculations[17] we have chosen two sample segments[18] of 13 obligors each with $\alpha = 65$ percent and $\alpha = 40$ percent, respectively. These are chosen randomly from Moody' Default Risk Service™ database of CCLs rated as of 31 December 2008. The segments are investment grade (rated Moody's Baa3 and higher) and junk grade (rated Moody's Ba1 and lower), respectively. Limits[19] for each exposure vary from \$25 MM to \$235 MM. The values of α are sourced from Jacobs (2010) based upon estimated additional drawdowns on unused limits (or "LEQ" factors) for Moody's rated CCLs 1987-2009 defaulting within a one-year horizon in Moody's Ultimate Recovery Database™. Each of the obligor's limit is divided into 1,000 puts each having a size of \$1,000. Figure AI in Appendix 2 shows that the usage distribution of each segment and the final convoluted portfolio; the descriptive statistics of each of the distributions are presented in Table AIII of Appendix 2 for reference. The convoluted distribution has both higher mean and higher standard deviation than either the segments. However, the distributional statistics reveal these to be near Gaussian, which we would like to overcome in future extensions of the model.

To explore the effect of choice of n in the exposure distribution we use a hypothetical five obligor portfolio A with the random unused limits. The portfolio and results of the experiment are summarized in Table AIII in Appendix 3. We note that the standard deviation of the usage distribution decreases as we increase the number of puts used. This may be explained by the fact that we are implicitly assuming a known value of α in our model (i.e. a zero volatility of α), and this feature becomes more prominent once we start increasing the number of puts n (i.e. as we approach a more realistic scenario).

The mean value remains relatively stable, but the extreme points converge towards the mean, producing a narrowing in the spread of the distribution.

Another prime variable in the algorithm is the value set for α, therefore we vary the value of α to study its effect on the final distribution. The results of this exercise are summarized in Table AIV in Appendix 3. For our five obligor portfolio, we see that an increase in the additional usage rate level also increases the volatility of the associated exposure distribution. In order to incorporate volatility of α explicitly in the model, we can also use a mixed Poisson process where we chose different values of α from an assumed distribution. A commonly used mixture distribution for α is the γ, resulting in negative binomial distribution for α, which has the advantage of being an analytically tractable two parameter distribution. We can use similar combinations[20] to incorporate volatility of α explicitly in each segment of the portfolio.

An argument against using any mixed distribution may be that this will induces a second set of assumptions in our model and will require banks to calculate the usage volatility of each segment. In the current model the volatility of the final distribution will depend how spread out the expected usage is between each the segment. This may be a more pragmatic approach considering that it will have minimal data requirements at the portfolio-segment level, and portfolio segmentation can be decided upon with the guidance of internal research and expert judgment on additional drawdown rates.

One portfolio segmentation possibility is based on commitment fees[21] and service fees[22]. As shown by Thakor and Udell (1987), when the bank is uncertain about the level of partial draw-down, it may segregate borrowers by keeping high commitment fees and low service fees in one contract, and low commitment fees and high service fees in another contract. The former would be attractive to borrowers with a higher probability of draw-down as they are more likely to pay a service fee and more interested in having an active credit line, while this will not be true for borrowers who are less confident about draw-downs. As discussed by Maksimovic (1990), contract choice may not be always that simple, since it may also depend upon structure of the borrower's industry; for instance, under imperfect competition the presence of predetermined rates of financing in a borrower's armory provides an enhanced strategic position.

V. Conclusion

This paper formulated a parsimonious model for the estimation of portfolio level EAD in a typical financial institution's CCL portfolio (or portfolio of similar exposures, such as reinsurance contracts for an insurer) by modeling each commitment with to an obligor as a portfolio of option instruments. We have modeled the exercise of each as a standard Poisson process where average usage (α) of CCLs at the portfolio level is assumed to be known. As indicated by several authors (Agarwal and Ambrose, 2006; Gatev and Strahan, 2006; Jacobs, 2010; Qi, 2009; Jiménez *et al.*, 2008), this value probably depends on changes in obligor credit quality, the outcome of the "race to default", the difference between contractual and market rates and other factors. Both empirical research and theory suggest a correlation between credit quality and usage of a credit line. The algorithm presented here can accommodate different values of α in order to model this correlation, as the portfolio may be segmented in terms of credit quality along with any other criterion decided upon by the financial institution. Various methods for estimating α have been outlined in the literature (Moral, 2006; Jacobs, 2010) and similar research is needed on expected additional usage rates for types of portfolios, such as analogous exposures to insurance companies as alluded to in this study. However, as discussed previously, these are likely to work best for financial institutions if this is based upon internal research, as results will be highly influenced by the firm's behavior in catching early signs of deterioration in credit quality. Further work may also be needed so that stable distribution parameters can be determined which will not be affected by choice of the number of puts used.

Most of the current credit risk models (e.g. Morgan's CreditMetrics, KMV Portfolio Manager, or CSFB CreditRisk + [23]) have a constant EAD as an input in calculating credit value-at-risk. Rosen and Sidelnikova (2002) find that stochastic exposures make a notable difference in economic credit capital calculations. Akkaya *et al.* (2003) analyze stochastic EAD in a CreditRisk + modeling framework, and they argue that stochastic EAD generated from such a model can be a useful in improvement in economic credit capital modeling for a CCL portfolio.

Furthermore, beyond economic capital applications, accurate expected exposure calculation is fundamental for liquidity risk management, for banks an example being a credit card portfolio or HELC where all the accounts are undrawn but committed lines, while for insures and example being property and casualty claims which be potential liabilities of unknown magnitudes. These pose a challenge to risk managers in terms of expected usage and henceforth liquidity positions, and the algorithm presented here may prove helpful in providing meaningful insight into the problem.

The other implication of the algorithm is in the area of EAD estimation for Basel II Advanced IRB (B2-AIRB) or Solvency II, in the case of banks or insurers, respectively. As pointed out earlier, as compared to PD estimation, limited research has gone into estimating EAD. Our contribution can provide a good starting point for financial institutions seeking to meet requirements under either of these supervisory regimes. Finally, this may also be used in terms of a stress-testing tool to determine worst case liquidity scenarios for portfolios including CCL like exposures, either for regulatory or internal risk management purposes. As we have the complete distribution of the usage values, we can get a good estimate of a worst-case scenario, from a high percentile depending upon the risk appetite of the institution.

There needs to be further work needs done to improve the algorithm in order to use it in standard software applications with minimized hardware requirements. This will greatly help in quick and smooth implementation of the otherwise intuitive model. We also would like to extend the statistical model to account for correlated exposures-at-default amongst exposures.

Notes

1. Refer to the BCBS (2009) revised document on the Basel II guidelines.
2. Searched on 24 February 2010 from www.google.com
3. Refer the "EAD expectation note" (FSA, 2007).
4. Refer to The European Commission on Banking Supervision (1998, 2002a, b, 2003a, b, 2004a, b, 2005) documents.
5. Contingent credit line, loan/credit commitment, line of credit have been used interchangeably in the literature. Other common terminologies in practice are "revolving lines of credit" or "unfunded commitments".

6. For details see Federal Deposit Insurance Corporation, Statistics on Banking, Table RC-6 (November 2009).
7. An example of regulatory modeling frameworks along these lines can be found in the standard formula developed by the German Insurance Association (2005). This model is an individualized market value and risk-based factor model that incorporates all controlling and monitoring-relevant risk categories, but focuses particularly on asset-liability mismatch risks, reinsurance default risks, and extreme events (e.g. resulting from natural hazards). Similar to the risk-based capital standards in the USA, the model includes interactions among these risk categories by using a root formula in aggregating different risk categories.
8. This is also referred as the LEQ in the literature (Araten and Jacobs, 2001; Jacobs, 2010).
9. Moral (2006) discusses that estimates of α are unstable when usage is very high.
10. Pricing of CCLs through an option theoretic approach has been used by Loukoianova *et al.* (2006), Chateau (1990) and Chateau *et al.* (2004).
11. In this case the bank is buying a put option from the obligor as the obligor is selling its debt to the bank by availing the credit lines on pre-specified terms and conditions.
12. They used a sample from the set of all commercial paper backup lines of credit for large US corporation in the period 1Q 1991-1Q 2002, for a total of 2,695 commitments.
13. This unit should be a positive integer, such as $1 million.
14. See the RMA Survey (2004) on the estimation of EAD and LGD.
15. For details on convolution using FFT please refer Melchiori (2004) and Robertson (1992).
16. See http://gmplib.org/ for details.
17. The calculations are done using Linux-based Genius 1.0.7 as an arbitrary precision calculator and Linux-based Octave for FFT and final distribution evaluation.
18. Details of the sample portfolio are presented in Appendix 1.
19. These obligors are chosen randomly from investment and junk ratings.
20. See Karlis and Xekalaki (2005) for various mixed Poisson distributions.
21. A commitment fee is an up-front fee paid when a commitment is made.
22. A service or annual fee is paid on the borrowed amount when the CCL is drawn upon.
23. For details see Gordy (2000) or Crouhy *et al.* (2000), which offers r comparisons of some the popular credit risk models in use.

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Appendix 1

Issuer number	Issuer name	Limit ($ 000)	Put size ($ ′000)	Moody's senior unsecured credit rating	Moody's broad industry category
153000	Central Maine Power Company	50,000	50	Baa1	Public utility
191670	Commercial Metals Company	235,000	235	Baa2	Industrial
232000	Detroit Edison Company (The)	68,750	69	Baa1	Public utility
252000	Duquesne Light Company	100,000	100	Baa2	Public utility
404000	Indianapolis Power & Light Company	120,600	121	Baa2	Public utility
490000	Michigan Consolidated Gas Company	81,250	82	A3	Public utility
576000	Orange and Rockland Utilities, Inc.	100,000	100	Baa1	Public utility
687000	South Carolina Electric & Gas Company	75,000	75	Baa1	Public utility
769000	Tucson Electric Power Company	120,000	120	Baa3	Public utility
600045390	IDACORP, Inc.	250,000	250	Baa2	Public utility
600050191	Rayonier Forest Resources, L.P.	50,000	50	Baa3	Real estate finance
600064222	Michigan Electric Transmission Company, LLC	25,000	25	Baa1	Industrial
808653810	NASDAQ OMX Group, Inc. (The)	150,000	150	Baa3	Securities

Note: $\alpha = 65$ percent

Table AI. Portfolio investment grade

Appendix 2

Issuer number	Issuer name	Limit ($ 000)	Put size ($ '000)	Moody's senior unsecured credit rating	Moody's broad industry category
600040059	Accuride Corporation	212,000	212	C	Industrial
809883143	Peach Holdings, Inc.	35,000	35	C	Finance
820360433	Bravo Health, Inc.	25,000	25	B2	Insurance
199515	Conseco, Inc.	80,000	80	Caa3	Insurance
600058771	GSCP (NJ), L.P.	60,000	60	C	Other non-bank
566800	Covanta Energy Corporation	100,000	100	Ba3	Public utility
807760066	Interstate Operating Company, L.P.	140,000	140	Caa3	Real estate finance
820399560	TPG-Austin Portfolio Holdings LLC	100,000	100	Ca	Real estate finance
431200	Kansas City Southern Railway Company (The)	100,000	100	B2	Transportation
809492974	Standard Steel, LLC	20,000	20	Caa1	Transportation
600038850	AEP Industries, Inc.	100,000	100	B2	Industrial
600042238	Alliance Laundry Systems LLC	230,000	230	B3	Industrial
44000	American Greetings Corporation	75,000	75	B1	Industrial

Note: $\alpha = 40$ percent

Table AII.
PORTFOLIO junk grade

Appendix 3

Portfolio	Investment grade	Junk grade	Convoluted portfolio (C)
α (%)	65.00	40.00	
Mean ('000 $)	926,640	510,800	1,437,400
SD ('000 $)	11,735	8,374	14,417
Skewness	0.015698	0.020199	0.012426
Kurtosis	3.0003	3.0005	3.00

Note: [a]Here n is set at 1,000

Table AIII.
Descriptive statistics of sample portfolio[a] limits

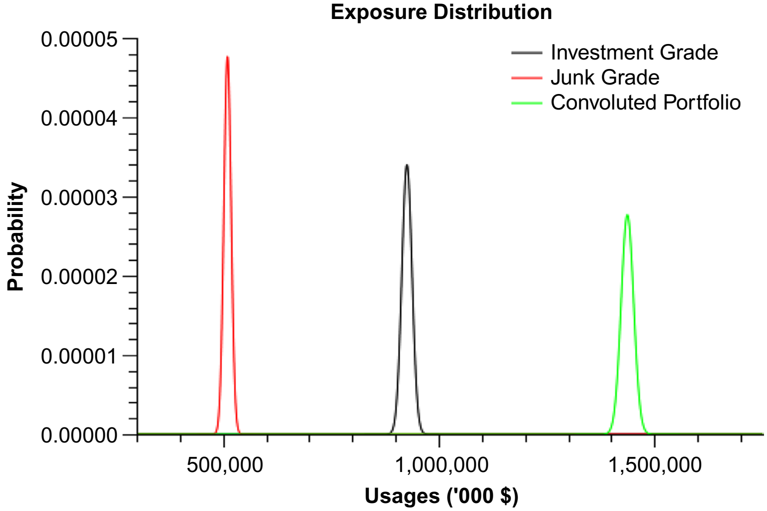

Figure A1. Chart for probability distribution of investment grade and junk portfolio

Table AIV. Variation of usage distribution ($) parameters with n [a]

Percentile	$n = 700$	$n = 800$	$n = 900$	$n = 1{,}000$	$n = 1{,}100$	$n = 1{,}200$	$n = 1{,}300$	$n = 1{,}400$	$n = 1{,}500$
50.00%	14,718	14,720	14,722	14,723	14,724	14,726	14,726	14,726	14,727
99.00%	17,272	17,100	16,970	16,849	16,741	16,662	16,583	16,520	16,460
99.50%	17,557	17,364	17,219	17,084	16,965	16,874	16,788	16,718	16,651
99.75%	17,823	17,612	17,452	17,304	17,173	17,074	16,978	16,903	16,829
99.90%	18,151	17,916	17,739	17,574	17,429	17,320	17,214	17,130	17,048
SD	1,058	988	936	886	842	809	777	751	726

Note: [a]Here α is kept constant at 10 percent

Table AV. Variation of usage distribution ($) parameters with α [a]

Percentile	$\alpha = 0.10$	$\alpha = 0.20$	$\alpha = 0.40$	$\alpha = 0.50$	$\alpha = 0.60$	$\alpha = 0.70$	$\alpha = 0.80$
50.00%	14,724	29,459	58,929	73,664	88,400	103,135	117,870
95.00%	16,214	31,552	61,876	76,955	92,001	107,023	122,024
99.00%	16,850	32,439	63,116	78,338	93,513	108,652	123,764
99.90%	17,575	33,445	64,519	79,901	95,220	110,492	125,728
99.97%	17,904	33,899	65,151	80,603	95,987	111,319	126,610
SD	886	1,253	1,722	1,981	2,170	2,344	2,506

Notes: [a]Here n is kept constant at 1,000; lower numbers for n and ⟨ is chosen to reduce computational time taken

Table AVI.
Hypothetical portfolio A

Obligor	Unused limit ($)	Actual put size	Rounded put size
A	81,289	81.29	82
B	13,626	13.63	14
C	10,941	10.94	11
D	20,709	20.71	21
E	20,786	20.79	21

Notes: $\alpha = 10$ percent; $n = 1{,}000$

Corresponding author
Michael Jacobs Jr can be contacted at: michael.jacobs@occ.treas.gov