

# Asset Price Bubbles and the Quantification of Credit Risk Capital

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## Abstract

This study presents an analysis of the impact of asset price bubbles on standard credit risk measures, including *Expected Loss* (“EL”) and *Credit Value-at-Risk* (“CVaR”). We present a stylized model of asset price bubbles in continuous time, and perform a simulation experiment of a 2 dimensional *Stochastic Differential Equation* (“SDE”) system for asset value determining *Probability of Default* (“PD”) through a *Constant Elasticity of Variance* (“CEV”) process, as well as a correlated a *Loss-Given-Default* (“LGD”) through a mean reverting *Cox-Ingersoll-Ross* (“CIR”) process having a long-run mean dependent upon the asset value. Comparing bubble to non-bubble economies, it is shown that asset price bubbles may cause an obligor’s traditional credit risk measures, such as EL and CVaR to decline, due to a reduction in the right skewness of the credit loss distribution. We propose a new risk measure in the credit risk literature to account for losses associated with a bubble bursting, the *Expected Holding Period Credit Loss* (“EHPCL”). We present evidence that asset price bubbles are a phenomenon that must be taken into consideration in the proper determination of economic capital for both credit risk management and measurement purposes. We also perform a sensitivity analysis of the SDE parameters upon the resulting credit risk measures, as well as the changes in their relationship to the CEV parameter, illustrating an application of an important model validation procedure.

Keywords: Financial Crisis, Credit Risk, Model Risk, Asset Price Bubbles, Expected Loss, Credit Value-at-Risk, Stochastic Differential Equations, Probability of Default, Loss Given Default, Constant Elasticity of Variance, Cox-Ingersoll-Ross.

JEL Classifications: C15, E58, G12, G13, G17, G18, G21, G28, G33.

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# 1 Introduction and Motivations

The financial crisis of the last decade has been the impetus behind a movement to better understand the relative merits of various risk measures, classic examples being *Value-at-Risk* (“VaR”) and related quantities (Jorion, 2006; Inanoglu and Jacobs, 2009). The importance of an augmented comprehension of these measures is accentuated in the realm of credit risk, as the price bubble in the housing market and the ensuing credit crunch, was undoubtedly a catalyst for the financial crisis. We have subsequently learned from this that the credit models of that era failed in not incorporating the phenomenon of price bubbles, which in turn added to the severity of the downturn for investors and risk managers who mis-measured their potential adverse exposure to credit risk. This manifestation of *model risk* (The U.S. Board of the Governors Federal Reserve System, SR 11-7), wherein a modeling framework lacks a key element of an economic reality and therefor fails, was due to some extent to a lack of basic understanding. This failure of the modeling paradigm in credit risk spans gaps in the measurement, characterization and economics of asset price bubbles.

In view of analyzing the impact of asset price bubbles on credit risk measures and credit capital determination with respect to a firm subject to default risk (or an investor holding a defaultable instrument), we construct various hypothetical economies, having and also not having asset price bubbles. In a stylized structural credit model framework (Merton, 1974), we simulate a firm’s value process in each of these economies, computing the firm’s standard risk measures. We present a model of asset price bubbles in continuous time, and perform a simulation experiment of a 2 dimensional *Stochastic Differential Equation* (“SDE”) system for asset value determining *Probability of Default* (“PD”)<sup>2</sup> through a *Constant Elasticity of Variance* (“CEV”) process<sup>3</sup>, as well as a correlated a *Loss-Given-Default* (“LGD”)<sup>4</sup> through a mean reverting *Cox-Ingersoll-Ross* (“CIR”; Cox et al, 1985) process having a long-run mean dependent upon the asset value. Comparing bubble to non-bubble economies, it is shown that asset price bubbles may cause an obligor’s traditional credit risk measures such as *Expected Loss* (“EL”) and *Credit Value-at-Risk* (“CVaR”) to decline, due to a reduced right skewness of the credit loss distribution. We propose a new risk measure in the credit risk literature to account for losses associated with a bubble bursting, the *Expected Holding Period Credit Loss* (“EHPCL”), a phenomenon that must be taken into consideration for the proper determination of economic capital for both credit risk management and measurement purposes.

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<sup>2</sup> For a Bayesian stochastic model for PD in the Basel *asymptotic single risk factor* (“ASRF”) class of models, underlying the Basel II Advanced IRB model for credit loss, see Jacobs and Kiefer (2010).

<sup>3</sup> For applications of the CEV model in finance see Chan et al (1992) and Jacobs (2001), in the context of term structure and interest rate derivatives.

<sup>4</sup> See Jacobs (2011, 2012) for a 2-factor structural credit model with stochastic LGD; Araten and Jacobs (2004) or Jacobs and Karagozolu (2011) for empirical models of LGD; and Frye and Jacobs (2012) for a structural credit risk model in ASRF class of models, featuring co-monotonic LGD and PD that provides a parsimonious function for downturn LGD.

The results of our experiment demonstrate that the existence of an asset price bubble, which occurs for certain parameter settings in the CEV model, results in the firm asset value distribution having both a greater right skewness. This augmented right skewness of a firm's return due to bubble expansion results in a reduction of the right skewness in the distribution of the default rate and a lower PD, which in combination with a lower mean of the LGD process, results in a credit loss distribution having lower right skewness. This in turn results in the decline of the firm's CVaR risk measures, an understatement in the credit risk of the firm. Based on these measures alone, their declining values imply that in the presence of asset price bubbles, less credit capital is required. However, as shown by the additional risk measure proposed in the present paper, the EHPCL, this conclusion is incorrect. This credit loss measure increases in bubble economies and is due to bubble bursting, which causes significant firm value losses, with accompanying credit losses on the bubble-bursting paths.

As asset price bubbles are inevitably bound to burst, causing significant credit loss to creditors, more credit capital should be held for these bubble-bursting scenarios. Unfortunately, the severity of these bubble-bursting scenarios is not adequately captured by the standard credit risk measures, whose computation is based on the standard moments and quantiles of a firm's credit loss distribution over time horizons such as a year, over which bubble bursting is unlikely. These bubble-bursting scenarios are captured, however, in some correctly constructed credit risk measures such as the EHPCL.

An outline for this paper is as follows. Section 2 presents a review of the literature. Section 3 presents our credit model incorporating the effect of asset price bubbles. Section 4 describes the results of our simulation experiment, while Section 5 summarizes the implications of our analysis for credit risk management.

## 2 Review of the Literature

Modern credit risk modeling (e.g., Merton, 1974) increasingly relies on advanced mathematical, statistical and numerical techniques to measure and manage risk in credit portfolios. This gives rise to *Model Risk* (The U.S. Board of Governors of the Federal Reserve System, 2011), defined as the potential that a model used to assess financial risks does not accurately capture those risks, and the possibility of understating inherent dangers stemming from very rare yet plausible occurrences perhaps not in reference data-sets or historical patterns of data<sup>5</sup>, a key example of this being the inability of the credit risk modeling paradigm to accommodate the phenomenon of asset price bubbles.

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<sup>5</sup> In the wake of the financial crisis (Demirguc-Kunt et al, 2010; Acharya et al, 2009), international supervisors have recognized the importance of *Stress Testing* ("ST"), especially in the realm of credit risk, as can be seen in the revised Basel framework (BCBS 2005, 2006; 2009 a,b 2010) and the Federal Reserve's *Comprehensive Capital Analysis and Review* ("CCAR") program (Jacobs 2013, Jacobs et al 2015).

The relative merits of various risk measures, classic examples being *Value-at-Risk* (“VaR”) and related quantities, have been discussed extensively by prior research (Jorion 1997, 2006). Risk management as a discipline in its own right, distinct from either general finance or financial institutions, is a relatively recent phenomenon. A general result of mathematical statistics due to Sklar (1956), allowing the combination of arbitrary marginal risk distributions into a joint distribution while preserving a non-normal correlation structure, readily found an application in finance. Among the early academics to introduce this methodology is Embrechts et al. (1999, 2002, 2003). This was applied to credit risk management and credit derivatives by Li (2000). The notion of copulas as a generalization of dependence according to linear correlations is used as a motivation for applying the technique to understanding tail events in Frey and McNeil (2001). This treatment of tail dependence contrasts to Poon et al (2004), who instead use a data intensive multivariate extension of extreme value theory, which requires observations of joint tail events.

Since the 2007 crisis, the mathematical finance literature has made significant advances in the modeling and testing of asset price bubbles (Jarrow and Protter, 2010; Hong et al, 2006). Inanoglu and Jacobs (2009) contribute to the modeling effort by providing tools and insights to practitioners and regulators, utilizing data from major banking institutions’ loss experience, exploring the impact of business mix and inter-risk correlations on total risk, and comparing alternative established frameworks for risk aggregation on the same data-sets across banks. Protter (2011), Protter et al (2010) and Jarrow et al (2014, 2015) apply these new insights to determine the impact that asset price bubbles have on the common risk measures used in practice for the determination of equity capital, which we extend to the realm of credit risk.

### 3 A Credit Model for Asset Price Bubbles

We model the evolution of asset prices, incorporating the phenomenon of price bubbles, using the approach of Jarrow et al (2007, 2014a, 2014b). The setting is a continuous trading economy, without loss of generality having a finite horizon  $[0, \tau]$ , with randomness described by the filtered probability space  $(\Omega, \mathfrak{F}, \mathbb{P})$ , where we define: the state space  $\Omega$ , the  $\sigma$ -algebra  $\mathfrak{F}$ , the information partition  $\mathbb{F} = \{\mathfrak{F}_t\}_{t \in [0, \tau]}$ , and the *physical probability measure*  $\mathbb{P}$  (or *actuarial*, as contrasted to a *risk-neutral probability measure*, commonly denoted by the symbol  $\mathbb{Q}$ ). We assume, again without loss of generality and for the purpose on focusing on the application to credit loss, a single asset value process  $\{V_t\}_{t \in [0, \tau]}$  that is adapted to the filtration  $\mathbb{F}$ . Note that this could also represent a share of stock owned by a representative equity investor, which is a claim on the single productive entity or firm in this economy. In the general setting,  $V_t$  follows an *Ito diffusion process* (Øksendal, 2003) having the following SDE representation:

$$dV_t = \mu(V_t, t)dt + \sigma(V_t, t)dW_t^L \quad (3.1)$$

where  $\mu(V_t, t)$  is the *instantaneous drift process*,  $\sigma(V_t, t)$  is the *instantaneous diffusion process*,  $W_t^L \sim N(0, t)$  is a *standard Weiner process* (or a *Brownian motion process*) on the filtered probability space  $(\Omega, \mathfrak{F}, \mathbb{P})$ , and  $dW_t$  are its infinitesimal increments. In order to complete this economy, we assume that there exists a traded *money market account process*  $M_t$ , which grows according to a risk-free rate process  $r_t$ , the latter also adapted to the filtration  $\mathbb{F}$  of the aforementioned probability space:

$$M_t = \exp\left\{\int_{s=0}^t r_s ds\right\}. \quad (3.2)$$

Without loss of generality we assume that the asset  $V_t$  has no cashflows, which could have been incorporated into the model by assuming a dividend process and studying the dividend-reinvested stock price process (Back, 2010), but to maintain simplicity of notation we do not do so.

We model an economy potentially having an price bubbles through the assumption that the risky asset's prices follows a *Constant Elasticity of Variance* (CEV) process, as in Jarrow et al (2014b), which is the following restricted version of the Ito diffusion process in equation (3.1):

$$dV_t = \mu V_t dt + \sigma V_t^\theta dW_t^V, \quad (3.3)$$

where  $\mu$  is the drift,  $\sigma$  is the volatility and the CEV parameter  $\theta$  governs the state of the risky asset price process exhibiting a price bubble or not. An asset price bubble is defined as the situation where the market price for an asset exceeds its *fundamental value* (Jarrow et al, 2007, 2010), the latter being defined conventionally price an investor would pay to hold the asset perpetually without rebalancing. This fundamental value is determined through the imposing some additional structure on the economy, requiring at minimum two additional assumptions. First, we need to assume that the absence of any arbitrage opportunities (Delbaen and Schachermayer, 1998), which guarantees the existence of a risk-neutral probability  $\mathbb{Q}$  measure equivalent to  $\mathbb{P}$ , such that the asset value process  $V_t$  normalized by the money market account  $M_t$  is a *local Martingale* process:

$$E^{\mathbb{Q}} \left[ \frac{V_{t'}^{\tau^*}}{M_{t'}} \middle| \mathfrak{F}_{t'} \right] = \left[ \frac{V_{(t, \tau^*)^-}}{M_t} \right] = V_t \quad \forall t' < t, \quad (3.4)$$

where  $V_t^{\tau^*} \equiv V_{(t, \tau^*)^-}$  is the *stopped process* of  $V_t$  and  $\tau^*: \Omega \rightarrow [0, +\infty)$  is a sequence of *stopping times* that satisfy certain technical condition.<sup>6</sup> The mechanism in (3.4) involving the risk-neutral probability measure affords us a means of computing present values where we shift the mass of the probability distribution (magnitude of the cash-flows) such that we can recover the same prices as under actuarial measure with the original cash-flows – but note that  $\mathbb{Q}$  is arbitrary. In order to pin

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<sup>6</sup> The conditions are that  $\tau^*$  is *almost surely increasing*  $P^{\mathbb{Q}}[\tau_k^* < \tau_{k+1}^*] = 1$  and is *almost surely divergent*

$P^{\mathbb{Q}}[\tau_k^* \rightarrow \infty \text{ as } k \rightarrow \infty] = 1$  (Oksendal, 2003).

down this risk-neutral distribution, we assume from this point on a *complete market*, which means that that enough derivatives on the risky assets trade in order to replicate its cash flows in a suitably constructed *arbitrage portfolio*. The first condition is satisfied because the CEV process given in expression (3.3) admits an equivalent local martingale measure, so by construction it satisfies the absence of arbitrage opportunities<sup>7</sup>. Under this incremental structure that we impose upon the economy, an asset's *fundamental value*  $FV_t$  given the time  $t$  information set  $\mathfrak{F}_t$ , is defined as the asset's discounted future payoff from liquidation at time at horizon  $\tau > t$ :

$$FV_t \left[ V_t \mid \mathfrak{F}_t \right] = E^Q \left[ \frac{V_\tau}{M_\tau} \mid \mathfrak{F}_t \right] M_t \quad (3.5)$$

It follows that we may define the asset's price bubble  $B_t^V[\bullet]$  as the difference between the market price  $V_t$  and its fundamental value  $FV_t$ :

$$B_t^V \left[ V_t \mid \mathfrak{F}_t \right] \equiv V_t - FV_t \left[ V_t \mid \mathfrak{F}_t \right]. \quad (3.6)$$

Since as a conditional expectation, the fundamental value normalized by the value of the money market account is a martingale under  $Q$ , a bubble exists if and only if the asset's normalized price is a strict local martingale and not a martingale under  $Q$ . In the case of the CEV process, it can be

shown (Jarrow et al, 2011) that the asset's normalized price  $\left( \frac{V_t}{M_t} \right)$  is a martingale under  $Q$  when

$\theta \leq 1$  in (3.3) (i.e., no asset price bubble), and a strict-local martingale under  $Q$  where  $\theta > 1$  in that equation (i.e., an asset price bubble). Note that the boundary case of  $\theta = 1$  yields the geometric Brownian motion underlying the *Black–Scholes–Merton* (“BSM”) option pricing model (Merton, 1974), which is called the *BSM economy*, and can be shown to exhibit no price bubble (Delbaen and Schachermayer, 1995).

In order to model the distribution of credit loss in the structural modeling paradigm (Merton, 1974), we define the unconditional loss process as an Ito process, similarly to the asset value process (3.1):

$$dL_t = \nu(L_t, V_t, t) dt + \varsigma(L_t, t) dW_t^L, \quad (3.7)$$

where  $\nu(L_t, t)$  is the *instantaneous drift process*,  $\varsigma(L_t, t)$  is the *instantaneous diffusion process*,  $W_t^L \sim N(0, t)$  is a standard Weiner process (or a Brownian motion process) on the filtered probability space  $(\Omega, \mathfrak{F}, F, P)$ , and  $dW_t^L$  are its infinitesimal increments. We assume that  $W_t^L$  is independent of  $W_t^V$ , but model the correlation between  $V_t$  and  $L_t$  through making the drift of the latter dependent on the former as follows:

$$dL_t = \nu_0(L_t - \nu_1 V_t) dt + \varsigma_0 L_t^{\varsigma_1} dW_t^L \quad (3.8)$$

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<sup>7</sup> This condition is sometimes termed “no free-lunch with vanishing risk” or NFLVR (Jeanblanc et al, 2009).

Therefore, for  $(\nu_0, \nu_1) \geq (0, 0)$  the linear drift function  $\nu(L_t, V_t, t) = \nu_0(L_t - \nu_1 V_t)$  of  $L_t$  is decreasing in  $V_t$ . Next, we develop the distribution of credit loss  $CL_\tau$  by defining an event of default as asset value  $V_t$  falling beneath the value of debt (or the default threshold)  $D$  the horizon  $\tau$  times the loss-given-default  $L_\tau$ :

$$CL_\tau \equiv I_{A_\tau < D} \times L_\tau. \quad (3.9)$$

We define expected credit loss  $ECL_\tau$  as the expectation of this random variable under actuarial probability measure  $P$ :

$$ECL_\tau = E^P [CL_\tau | \mathfrak{F}_t]. \quad (3.10)$$

We may estimate this quantity as  $ECL_\tau$  through numerical integration over  $N_p$  simulations, which is simply the sample mean, which is a consistent and unbiased estimator of this sample moment:

$$ECL_\tau = \frac{1}{N_p} \sum_{i=1}^{N_p} CL_\tau^i. \quad (3.11)$$

Similarly, we may obtain estimators of the population *standard deviation*  $\sigma_\tau^{ECL}$  and of the population *normalized skewness*  $\zeta_\tau^{ECL}$  of this distribution, defined as:

$$\sigma_\tau^{ECL} = \sqrt{E^P \left[ (CL_\tau - ECL_\tau)^2 | \mathfrak{F}_t \right]}, \quad (3.12)$$

$$\zeta_\tau^{ECL} = \frac{E^P \left[ (CL_\tau - ECL_\tau)^3 | \mathfrak{F}_t \right]}{(\sigma_\tau^{ECL})^{\frac{3}{2}}}, \quad (3.13)$$

by their sample analogues  $\hat{\sigma}_\tau^{ECL}$  and  $\hat{\zeta}_\tau^{ECL}$ :

$$\hat{\sigma}_\tau^{ECL} = \sqrt{\frac{1}{N_p - 1} \sum_{i=1}^{N_p} \left[ CL_\tau^i - \frac{1}{N_p} \sum_{i=1}^{N_p} CL_\tau^i \right]^2}, \quad (3.14)$$

$$\hat{\zeta}_\tau^{ECL} = \frac{\frac{1}{N_p} \sum_{i=1}^{N_p} \left[ CL_\tau^i - ECL_\tau \right]^3}{(\hat{\sigma}_\tau^{ECL})^{\frac{3}{2}}}. \quad (3.15)$$

These statistics are estimates of the credit loss distribution's moments under the physical probability measure  $P$ , characterizing the changes in the value of a defaultable instrument that includes both positive and negative mark-to-market values. In a credit risk management application, we are actually only interested in losses, to which end we seek to understand the right tail of the credit

loss distribution, and compute various *high quantile risk measures*, such as the VaR (in market risk) or CVaR (its analogue in credit risk). Apart from asset price bubbles, even though its limitations are widely known (Alexander 2001; Jorion 1997), such measures are widely used in the industry<sup>8</sup>. An estimator for the CVaR at a given confidence level  $c$  is given by:

$$CVaR_{\tau}(c) = \text{Quantile}_{i \leq k \leq N_p(c)}, \quad (3.16)$$

where

$$\text{Quantile}_{i \leq k \leq N_p(c)}(c) = \inf_x \left\{ \left[ \frac{1}{N_p} \sum_{k=1}^{N_p} I_{CL_{\tau}^k \geq x} \right] \geq c \right\}, \quad (3.17)$$

where  $I_{CL_{\tau}^k \geq x}$  is an indicator function that takes the value 1 if  $CL_{\tau}^k \geq x$  and 0 otherwise. We may also define a conditional CVaR measure, the *Expected Tail Credit Value-at-Risk* (“ETCVaR”), as the expected loss conditional on credit loss being greater than or equal to loss at confidence level  $c$ :

$$ETCVaR(c) = \frac{\sum_{k=1}^{N_p} CL_{\tau}^k I_{CL_{\tau}^k \geq CVaR(c)}}{\sum_{k=1}^{N_p} I_{CL_{\tau}^k \geq CVaR(c)}}. \quad (3.18)$$

In order to model the influence of an entire boom to bust cycle of asset values upon credit risk measures, we introduce a measure of credit loss over a sequence of smaller sub-intervals, which we call the *Expected Holding Period Credit Loss* (“EHPCL”). Let  $t_{HP}$  be a shorter holding period (e.g., 1-week),  $t_{Nd}$  be the longer (e.g., 1-year) horizon of the simulation and  $N_s = t_{Nd}/t_{HP}$  is the number of consecutive sub-intervals of length  $t_{HP}$  (e.g., 50 weeks). Then we define the  $t_{HP}$  day credit loss  $CL_{k,s}$  for sub-period  $s \in \{1, \dots, N_s\}$  along simulation path  $k \in \{1, \dots, N_p\}$  as a situation in which the credit loss is greater than the estimated expected loss over the simulation horizon  $ECL_{\tau}$ :

$$CL_{k,s} = CL_{\tau}^{k,s} I_{CL_{\tau}^{k,s} > ECL_{\tau}}. \quad (3.19)$$

Then we define the *EHPCL* as the average of this quantity over sub-intervals and over simulation paths:

$$EHPCL = \frac{1}{N_p N_s} \sum_{k=1}^{N_p} \sum_{s=1}^{N_s} CL_{k,s}. \quad (3.20)$$

Finally, we make note that we may look at all of these measures on a relative basis through scaling by the  $ECL_{\tau}$ , the estimators for which we omit for the sake of brevity.

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<sup>8</sup> This is partly due to the Basel II Accords (Engelmann and Rauhmeier (2006), Cornford (2005)).



## 4 A Simulation Experiment

We demonstrate the impact of asset price bubbles on an obligor’s credit risk measures, as defined in the previous section (i.e., EL, CVaR, ETCVaR and EHPCL), through a stochastic simulation experiment. Simulation is needed to determine the probability distribution of asset value, loss given default rate and ultimately the credit loss rate over a given time interval. In our experiment we fix the time period for the standard credit risk measures to be 250 trading days or one year, which is conventional for economic and regulatory credit risk capital calculations, with sub-intervals of one business week or five trading days for the EHPCL measure. We use simulation because an analytic solution for the firm value’s probability distribution using the CEV process is unavailable (Emanuel and MacBeth (1982), Schroder (1989))<sup>9</sup>

We perform the simulation experiment through constructing a collection of different economies, some with bubbles and some without, by varying the CEV parameter  $\theta$  from 0.25 to 2.0 in steps of size 0.25. In each of these different economies, we compute the standard risk measures to determine the impact that bubbles have on their values. We fix the other parameters of the simulation as follows. Asset value  $V_t$  is initiated at a normalized value of one  $V_0 = 1$ , with a drift rate of 5% per annum,  $\mu = 0.05$ , and a volatility parameter of 10% per annum,  $\sigma = 0.10$ . Default is assumed to occur if asset value at the horizon is below the debt threshold of 0.80,  $V_T \leq D = 0.80$ . The mean reversion parameter of the LGD process  $L_t$  is set to be  $\nu_0 = 0.80$ , and the sensitivity of the long-run mean to asset value is set to  $\nu_1 = 0.40$ , with an initial value  $L_0 = 0.40$ ; and a diffusion function having volatility of  $\varsigma_0 = 0.4$  and CEV parameter  $\varsigma_1 = 0.25$ .

The results of our analysis are tabulated in Tables 4.1 through 4.2. In Table 4.1 we present the absolute credit loss measures as defined in Section 3, whereas in Table 4.2 we present these measures on a relative basis, scale by the expected credit loss. We also show the estimate of PD and the distributional statistics of the asset value, LGD and credit loss processes. Figure 4.1 and 4.2 plot some key credit loss measures against the values of the CEV parameter, both in absolute and relative terms with respect to EL, respectively. In the Appendix Section 6, Figures 6.1 through 6.8 plot the distributions of asset value, LGD and credit loss processes, as well as the simulation paths of the asset value and LGD processes.

We can see from the results that all of the standard risk measures decrease as we increase the CEV parameter into the region where we have an asset price bubble, whereas the EHPCL increases monotonically and at an increasing rate as this parameter grows. We observe that that this is driven by a change in the asset value process as  $\theta$  rises, as we observe the skewness to rise monotonically and dramatically, doubling from 0.3089 to 0.6485 as we go from Geometric Brownian Motion (GBM) at  $\theta = 1$  to a bubble economy with  $\theta = 2$ . Note that other features of the asset value distribution are largely unchanged, such as the mean or the standard deviation, as we increase  $\theta$ . Note further that we see increasing asset value skewness in  $\theta$  even when in the non-bubble region  $\theta \in [0.25, 1.0]$ , with skewness about quadrupling from 0.08 to 0.31. On the other hand the LGD

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<sup>9</sup> We use the R package Sim.DiffProc to simulate the 2-dimensional system of SDEs for asset value and for LGD (R Development Core Team, 2015)

**Table 4.1: Alternative Absolute Credit Risk Loss Measures and Distributional Statistics – Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80, \sigma = 0.10$ ) and Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80, \nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) for Various Values of the CEV Parameter**

	Theta									
	0.25	0.5	0.75	1.00	1.25	1.5	1.75	2.00		
Probability of Default	0.6600%	0.5800%	0.3700%	0.3900%	0.3500%	0.1800%	0.2100%	0.0700%		
Minimum	61.61%	68.37%	70.84%	69.65%	72.37%	71.81%	71.77%	76.97%		
1st Quartile	97.98%	98.04%	97.94%	97.67%	97.73%	97.63%	97.33%	97.37%		
Median	105.00%	105.00%	104.70%	104.50%	104.60%	104.50%	103.90%	103.90%		
Mean	105.00%	105.20%	105.10%	105.10%	105.20%	105.20%	105.00%	111.70%		
3rd Quartile	111.90%	112.00%	111.80%	111.80%	112.10%	111.80%	111.60%	111.70%		
Asset Value Maximum	146.40%	148.50%	157.80%	158.50%	168.90%	176.90%	155.60%	170.30%		
Standard Deviation	10.37%	10.35%	10.32%	10.54%	10.77%	10.66%	10.89%	10.97%		
Skewness	0.0841	0.1224	0.2452	0.3089	0.3853	0.5212	0.5490	0.6485		
Minimum	-0.17%	3.38%	3.33%	2.32%	2.48%	3.50%	2.98%	0.06%		
1st Quartile	25.94%	27.50%	27.64%	27.17%	27.59%	27.47%	27.48%	26.30%		
Median	39.98%	38.06%	38.22%	37.94%	38.58%	38.32%	38.40%	40.09%		
Mean	55.81%	40.65%	40.62%	40.49%	40.85%	40.44%	40.66%	55.31%		
3rd Quartile	136.70%	51.08%	51.15%	51.01%	51.43%	50.96%	51.39%	159.80%		
Loss-Given Default Maximum	146.40%	150.60%	164.20%	143.40%	156.40%	140.30%	139.40%	159.80%		
Standard Deviation	21.72%	10.35%	17.87%	18.08%	18.01%	17.61%	17.96%	21.75%		
Skewness	0.5850	0.8456	0.7782	0.7978	0.8182	0.7664	0.7803	0.6425		
Minimum	0.0155%	0.0619%	0.0286%	0.0444%	0.0250%	0.0337%	0.0178%	0.0024%		
1st Quartile	0.1282%	0.1590%	0.0833%	0.0792%	0.0802%	0.0481%	0.0388%	0.0134%		
Median	0.2129%	0.2142%	0.1117%	0.1329%	0.1043%	0.0715%	0.0593%	0.0195%		
Mean	0.2387%	0.2492%	0.1313%	0.1412%	0.1235%	0.0725%	0.0609%	0.0178%		
3rd Quartile	0.3310%	0.2967%	0.1664%	0.2071%	0.1432%	0.0873%	0.0788%	0.0224%		
Credit Loss Processes Maximum	0.5429%	0.6146%	0.3595%	0.2660%	0.3781%	0.1305%	0.1324%	0.0309%		
Expected Credit Loss	0.2387%	0.2492%	0.1313%	0.1412%	0.1235%	0.0960%	0.0609%	0.0250%		
Credit VAR - 90th Percentile	0.4405%	0.4442%	0.2179%	0.2309%	0.2045%	0.0997%	0.0609%	0.0268%		
Credit Expected Shortfall - 90th Percentile	0.4939%	0.5215%	0.2844%	0.2510%	0.2891%	0.1169%	0.0896%	0.0309%		
Credit VAR - 99th Percentile	0.5367%	0.6110%	0.3344%	0.2309%	0.3573%	0.1259%	0.1284%	0.0305%		
Credit Expected Shortfall - 99th Percentile	0.5429%	0.6146%	0.3595%	0.2510%	0.3781%	0.1305%	0.1324%	0.0309%		
Credit VAR - 99.9th Percentile	0.5423%	0.6143%	0.3570%	0.2656%	0.3760%	0.1301%	0.1320%	0.0308%		
Credit Expected Shortfall - 99.9th Percentile	0.5429%	0.6146%	0.3595%	0.2660%	0.3781%	0.1305%	0.1324%	0.0308%		
Credit VAR - 99.97th Percentile	0.5427%	0.6145%	0.3588%	0.2659%	0.3775%	0.1304%	0.1323%	0.0309%		
Credit Expected Shortfall - 99.97th Percentile	0.5429%	0.6146%	0.3595%	0.2660%	0.3781%	0.1305%	0.1324%	0.0309%		
Standard Deviation of Credit Loss	0.1365%	0.1279%	0.0732%	0.0674%	0.0744%	0.0263%	0.0304%	0.0098%		
Skewness of Credit Loss	1.0522	0.8583	0.6448	0.4888	0.3239	0.2436	0.0849	-0.2660		
Expected Holding Period Credit Loss	0.3161%	0.3534%	0.4072%	0.4632%	0.5413%	0.6622%	0.7408%	0.8791%		

process does not have a very pronounced decrease in skewness as we increase  $\theta$ , and the pattern is non-monotonic, although from BGM to a CEV parameter of 2 LGD skewness decreases from 0.80 to 0.64, which is significant; and as with asset value, other features of the distribution are basically unchanged, including the mean, which is surprising in that the drift is a function of the asset value process. The effect of the asset value process can be seen in the average PD in each simulation, decreasing monotonically from 0.66% for  $\theta = 0.25$ , to 0.39% for  $\theta = 1.00$ , and finally to 0.07% for  $\theta = 2.00$ .

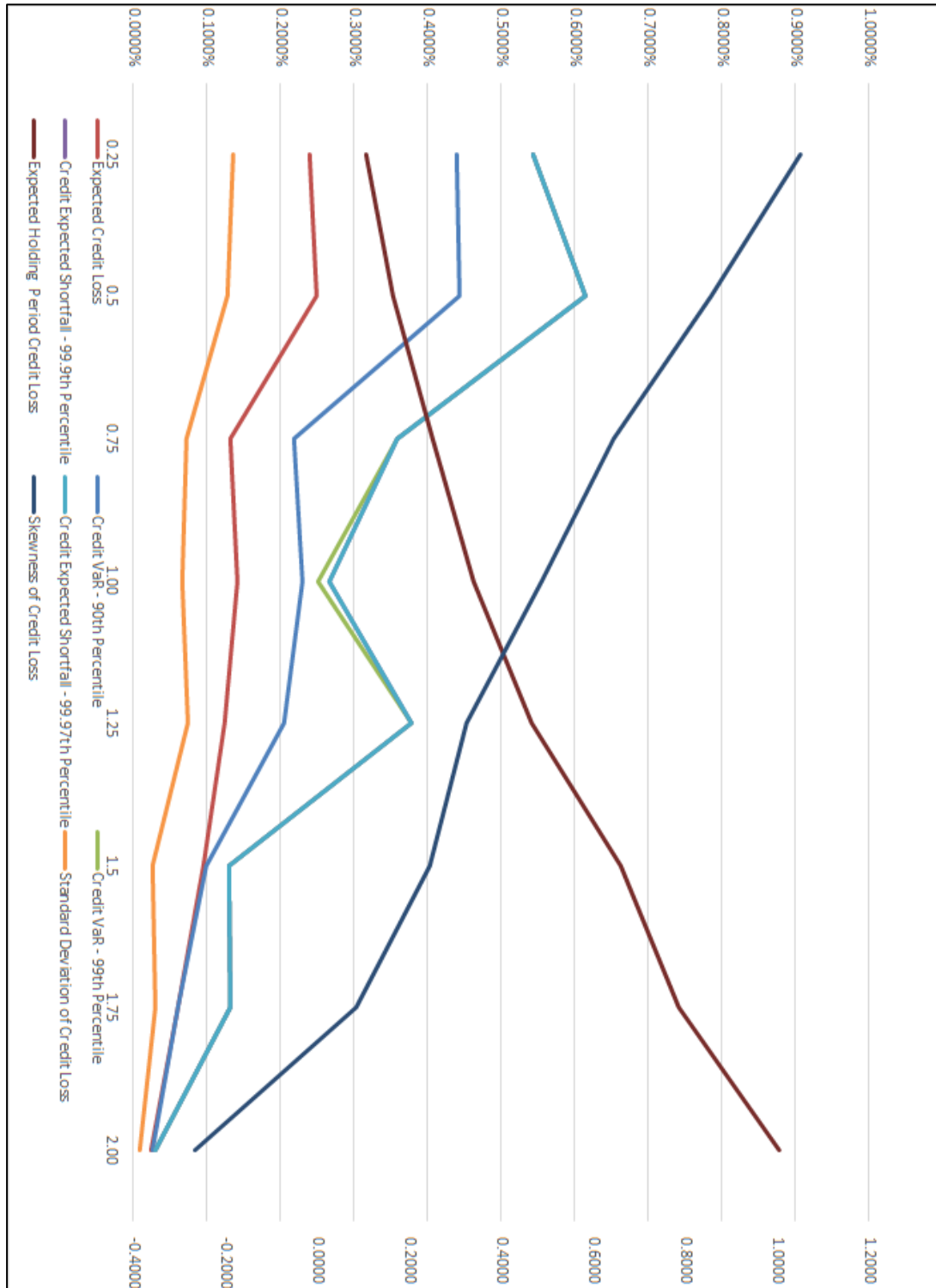
In the bottom panel of Table 4.1 we show the various credit loss metrics as a functions of the CEV parameter, and observe that all of the standard measures are declining in  $\theta$ . The ECL decreasing monotonically from 0.24% for  $\theta = 0.25$ , to 0.14% for  $\theta = 1.00$ , and finally to 0.03% for  $\theta = 2.00$ . All of these CVaR measures also decline monotonically and at an accelerating pace in  $\theta$ , and the steepness of the decline actually increases as the confidence level becomes more conservative. For instance, for the 99.97<sup>th</sup> percentile CVaR, it decreases monotonically and at an accelerating rate in  $\theta$  from 0.54% for  $\theta = 0.25$ , to 0.27% for  $\theta = 1.00$ , and finally to 0.03% for  $\theta = 2.00$ . The standard deviation of credit loss also decreases monotonically and at an accelerating rate from 0.14% for  $\theta = 0.25$ , to 0.07% for  $\theta = 1.00$ , and finally to 0.01% for  $\theta = 2.00$ . In contrast, the EHPCL *increases* monotonically and at an accelerating rate in  $\theta$  from 0.34% for  $\theta = 0.25$ , to 0.46% for  $\theta = 1.00$ , and finally to 0.88% for  $\theta = 2.00$ . This is reflected in the declining skewness of the credit loss distribution in  $\theta$  from 1.05 for  $\theta = 0.25$ , to 0.49 for  $\theta = 1.00$ , and finally to -27 for  $\theta = 2.00$  - implying that skewness this is a robust indicator of a bubble, but does not have the benefit of having an intuitive economic interpretation as does the EHPCL.

In the bottom panel of Table 4.2 we show the various relative credit loss metrics (scaled by ECL) as a functions of the CEV parameter, and observe that all of the standard relative risk measures are declining in  $\theta$ , as are the absolute versions in Table 4.1. All of three relative CVaR measures also decline monotonically and at an accelerating pace in  $\theta$ , and the steepness of the decline actually increases as the confidence level becomes more conservative. For instance, for the 99.97<sup>th</sup> percentile CVaR to ECL ratio, it decreases monotonically in  $\theta$  from 2.27 for  $\theta = 0.25$ , to 1.88% for  $\theta = 1.00$ , and finally to 1.23% for  $\theta = 2.00$ . The standard deviation to ECL ratio also decreases monotonically and at an accelerating rate from 0.57 for  $\theta = 0.25$ , to 0.47 for  $\theta = 1.00$ , and finally to 0.39 for  $\theta = 2.00$ . In contrast, the EHPCL to ECL ratio *increases* monotonically and at an accelerating rate in  $\theta$  from 1.33% for  $\theta = 0.25$ , to 3.28 for  $\theta = 1.00$ , and finally to 35.11 for  $\theta = 2.00$ . This is reflected in the declining ratio of skewness to ECL ratio in  $\theta$  from 440.72 for  $\theta = 0.25$ , to 346.17 for  $\theta = 1.00$ , and finally to -1062.11 for  $\theta = 2.00$ .

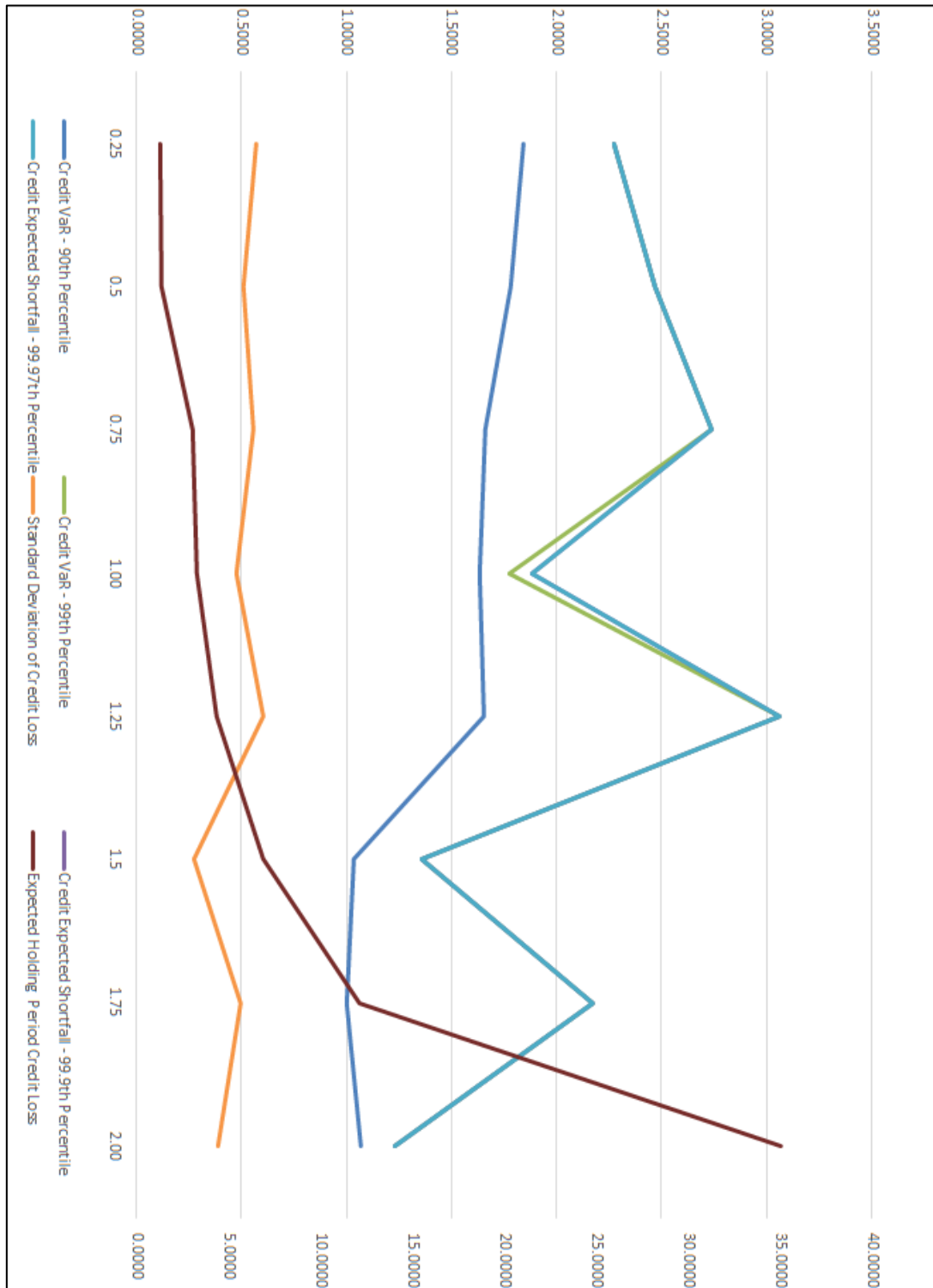
**Table 4.2: Alternative Relative Credit Risk Loss Measures and Distributional Statistics – Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80, \sigma = 0.10$ ) and Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80, \nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) for Various Values of the CEV Parameter**

	Theta									
	0.25	0.5	0.75	1.00	1.25	1.5	1.75	2.00		
Probability of Default										
Minimum	0.6600%	0.5800%	0.3700%	0.3900%	0.3500%	0.1800%	0.2100%	0.0700%		
1st Quartile	61.61%	68.37%	70.84%	69.65%	72.37%	71.81%	71.77%	76.97%		
Median	97.98%	98.04%	97.94%	97.67%	97.73%	97.63%	97.33%	97.37%		
Mean	105.00%	105.00%	104.70%	104.50%	104.60%	104.50%	103.90%	103.90%		
3rd Quartile	105.00%	105.20%	105.10%	105.10%	105.20%	105.20%	105.00%	111.70%		
Asset Value	111.90%	112.00%	111.80%	111.80%	112.10%	111.80%	111.60%	111.70%		
Maximum	146.40%	148.50%	157.80%	158.50%	168.90%	176.90%	155.60%	170.30%		
Standard Deviation	10.37%	10.35%	10.32%	10.54%	10.77%	10.66%	10.89%	10.97%		
Skewness	0.0841	0.1224	0.2452	0.3089	0.3853	0.5212	0.5490	0.6485		
Minimum	-0.17%	3.38%	3.33%	2.32%	2.48%	3.50%	2.98%	0.06%		
1st Quartile	25.94%	27.50%	27.64%	27.17%	27.59%	27.47%	27.48%	26.30%		
Median	39.98%	38.06%	38.22%	37.94%	38.58%	38.32%	38.40%	40.09%		
Mean	55.81%	40.65%	40.62%	40.49%	40.85%	40.44%	40.66%	55.31%		
3rd Quartile	136.70%	51.08%	51.15%	51.01%	51.43%	50.96%	51.39%	159.80%		
Loss-Given Default	146.40%	150.60%	164.20%	143.40%	156.40%	140.30%	139.40%	159.80%		
Standard Deviation	21.72%	10.35%	17.87%	18.08%	18.01%	17.61%	17.96%	21.75%		
Skewness	0.5850	0.8456	0.7782	0.7978	0.8182	0.7664	0.7803	0.6425		
Minimum	0.0155%	0.0619%	0.0286%	0.0444%	0.0250%	0.0337%	0.0178%	0.0024%		
1st Quartile	0.1282%	0.1590%	0.0833%	0.0792%	0.0802%	0.0481%	0.0388%	0.0134%		
Median	0.2129%	0.2142%	0.1117%	0.1329%	0.1043%	0.0715%	0.0593%	0.0195%		
Mean	0.2387%	0.2492%	0.1313%	0.1412%	0.1235%	0.0725%	0.0609%	0.0178%		
3rd Quartile	0.3310%	0.2967%	0.1664%	0.2071%	0.1432%	0.0873%	0.0788%	0.0224%		
Credit Loss Processes	0.5429%	0.6146%	0.3595%	0.2660%	0.3781%	0.1305%	0.1324%	0.0309%		
Expected Credit Loss	0.0024	0.0025	0.0013	0.0014	0.0012	0.0010	0.0006	0.0003		
Credit VaR - 90th Percentile	1.8451	1.7825	1.6592	1.6351	1.6549	1.0384	1.0000	1.0690		
Credit Expected Shortfall - 90th Percentile	2.0687	2.0927	2.1655	1.7772	2.3399	1.2177	1.4709	1.2324		
Credit VaR - 99th Percentile	2.2481	2.4520	2.5461	1.6351	2.8918	1.3113	2.1070	1.2160		
Credit Expected Shortfall - 99th Percentile	2.2740	2.4665	2.7378	1.7772	3.0605	1.3594	2.1728	1.2324		
Credit VaR - 99.9th Percentile	2.2714	2.4651	2.7187	1.8807	3.0436	1.3546	2.1662	1.2307		
Credit Expected Shortfall - 99.9th Percentile	2.2740	2.4665	2.7378	1.8836	3.0605	1.3594	2.1728	1.2307		
Credit VaR - 99.97th Percentile	2.2732	2.4661	2.7321	1.8827	3.0554	1.3580	2.1708	1.2319		
Credit Expected Shortfall - 99.97th Percentile	2.2740	2.4665	2.7378	1.8836	3.0605	1.3594	2.1728	1.2324		
Standard Deviation of Credit Loss	0.5718	0.5132	0.5574	0.4772	0.6026	0.2743	0.4986	0.3912		
Skewness of Credit Loss	440.7221	344.4569	491.0177	346.1735	262.2027	253.7112	139.3435	-1062.1052		
Expected Holding Period Credit Loss	1.3238	1.4181	3.1008	3.2804	4.3812	6.8967	12.1572	35.0964		

**Figure 4.1: Alternative Absolute Credit Risk Loss Measures– Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80, \sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default ( $\nu_0 = 0.80, \nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes or Various Values of the CEV Parameter**



**Figure 4.2: Alternative Relative Credit Risk Loss Measures – Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80, \sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default ( $\nu_0 = 0.80, \nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes for Various Values of the CEV Parameter**



In Table 4.3, and in Figures 4.11 through 4.14, we present a sensitivity analysis of our model, across a range of setting for the CEV parameter. The sensitivities considered are as follows:

- Increase (decrease) asset value process volatility ( $\sigma$ ) to 20% (8%)
- Increase (decrease) asset value process drift ( $\mu$ ) to 7% (3%)
- Increase (decrease) LGD process volatility ( $\zeta_0$ ) to 60% (20%)
- Increase (decrease) LGD process drift ( $\nu_0$ ) to 7% (9%)
- Increase (decrease) default threshold ( $c$ ) to 90% (75%)

First, let us consider the effect on the 99.97<sup>th</sup> percentile CVaR (CVaR-99.97) and on the EHPECL of shocking asset return volatility  $\sigma$ , which is shown in the 1<sup>st</sup> two panels of Table 4.3. We observe that when  $\sigma$  is increased (decreased) to 20% (8%), as expected the CVaR99.97 and EHPECL both increase (decrease) from the base case, which is intuitive in that greater (lesser) asset value volatility implies an increased (decreased) probability of default. Note that the risk measures CVaR99.97 and EHPECL generally change in greater proportion to the change in the asset volatility parameter across values of the CEV parameter  $\theta$ , with the CVaR99.97 and EHPECL increasing (decreasing) about on the order of 80-fold and 10-fold (100% for both) for a doubling (20% decline) in  $\sigma$ . Furthermore, note that we observe some asymmetries in the change in the relationship between the risk measures and the CEV parameter  $\theta$ , in that we as we increase (decrease)  $\sigma$  the rate of decrease of the CVaR-99.97 in  $\theta$  is higher (about the same) as compared to the base. For example, for an increase (decrease) in  $\sigma$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in CVaR-99.97 are about  $[1800\%, 4200\%]$  ( $[-92\%, -88\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in range in the CVaR-99.97 are about  $[2800\%, 7000\%]$  ( $[-98\%, -92\%]$ ). On the other hand, when we increase or decrease  $\sigma$ , we observe a symmetry in the EHPECL, as in both cases it is increasing in the CEV parameter  $\theta$ , albeit at a lower rate shocking in either direction. For example, for an increase (decrease) in  $\sigma$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in EHPECL are about  $[1200\%, 1500\%]$  ( $[-93\%, -82\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes are in the EHPECL are about in the ranges  $[1000\%, 1200\%]$  ( $[-85\%, -50\%]$ ).

Secondly, let us consider the effect on the CVaR-99.97 and on the EHPECL of shocking asset return drift  $\mu$ , which is shown in the 3<sup>rd</sup> and 4<sup>th</sup> panels of Table 4.3. We observe that when  $\mu$  is increased (decreased) to 7% (3%), as expected the CVaR99.97 and EHPECL both decrease (increase) from the base case, which is intuitive in that greater (lesser) asset value drift implies an increased (decreased) probability of default. Note that the risk measures CVaR99.97 and EHPECL generally change in greater proportion to the change in the asset volatility parameter across values of the CEV parameter  $\theta$ , with the CVaR99.97 and EHPECL decreasing (increasing) about on the order of 40% and 60-fold (2-fold and 4-fold) for a 40% increase (decline) in  $\mu$ . Furthermore, note that we observe some asymmetries in the change in the relationship between the

**Table 4.2: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Credit Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 7\}$ )**

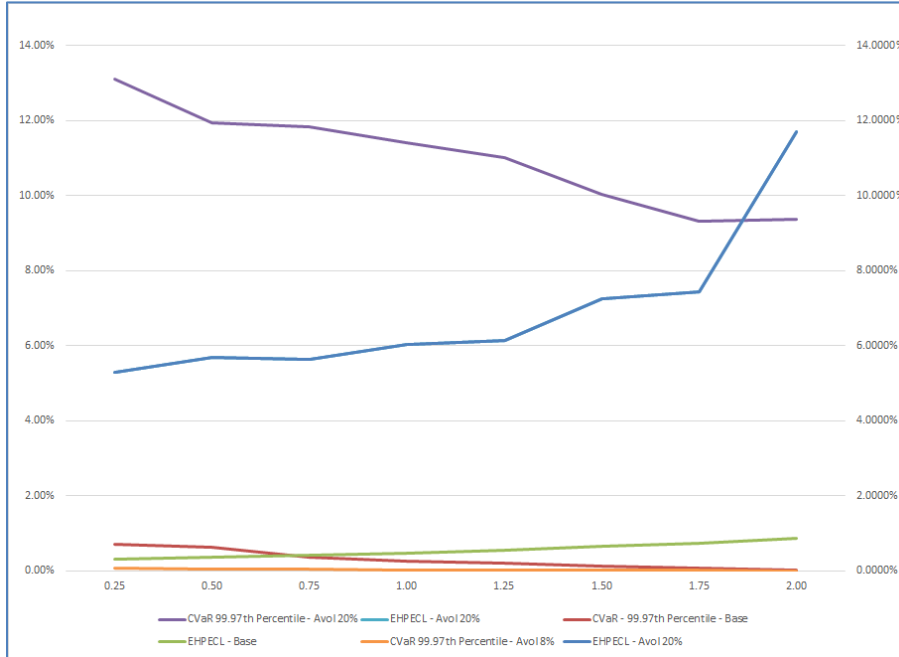
		Theta							
		0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Base Case	CVaR - 99.97th Percentile - Base	0.7055%	0.6145%	0.3588%	0.2659%	0.2097%	0.1304%	0.0661%	0.0309%
	EHPCL - Base	0.3161%	0.3534%	0.4072%	0.4632%	0.5413%	0.6622%	0.7408%	0.8791%
Increase Asset Volatility to 20%	CVaR 99.97th Percentile - Avol 20%	13.10%	11.95%	11.83%	11.42%	11.00%	10.04%	9.31%	9.38%
	Percent Change from Base	1756.20%	1843.95%	3197.62%	4194.68%	5147.40%	7597.58%	13972.59%	30284.78%
Decrease Asset Volatility to 8%	CVaR 99.97th Percentile - Avol 8%	0.0660%	0.0497%	0.0483%	0.0225%	0.0068%	0.0051%	0.0095%	0.0024%
	Percent Change from Base	-90.64%	-91.91%	-86.54%	-91.55%	-96.75%	-96.08%	-85.57%	-92.27%
Increase Asset Drift to 7%	CVaR 99.97th Percentile - Adrift 7%	0.3432%	0.2090%	0.1636%	0.1532%	0.1168%	0.1069%	0.0971%	0.0305%
	Percent Change from Base	-51.36%	-65.99%	-54.39%	-42.38%	-44.32%	-17.99%	46.81%	-1.27%
Decrease Asset Drift to 3%	CVaR 99.97th Percentile - Adrift 3%	1.0379%	0.8073%	0.8143%	0.8214%	0.5519%	0.3101%	0.3197%	0.2651%
	Percent Change from Base	47.12%	31.36%	126.97%	208.97%	163.17%	137.80%	383.32%	759.32%
Increase LGD Volatility to 6%	CVaR 99.97th Percentile - Lvol 6%	0.8888%	0.7342%	0.6260%	0.5178%	0.4660%	0.2200%	0.1860%	0.0465%
	Percent Change from Base	25.97%	19.48%	74.48%	94.76%	122.21%	68.74%	181.19%	50.69%
Decrease LGD Volatility to 2%	CVaR 99.97th Percentile - Lvol 2%	0.4571%	0.3370%	0.3012%	0.2578%	0.1779%	0.1289%	0.0732%	0.0176%
	Percent Change from Base	-35.21%	-45.16%	-16.04%	-3.04%	-15.15%	-1.16%	10.70%	-43.10%
Increase LGD Drift to 90%	CVaR 99.97th Percentile - Ldrft 90%	0.7330%	1.0679%	0.7322%	0.4766%	0.3489%	0.2727%	0.1758%	0.0789%
	Percent Change from Base	3.89%	73.78%	104.08%	79.27%	66.35%	109.13%	165.81%	155.77%
Decrease LGD Drift to 70%	CVaR 99.97th Percentile - Ldrft 70%	0.6961%	0.3652%	0.1844%	0.1254%	0.0933%	0.0509%	0.0420%	0.0024%
	Percent Change from Base	-1.33%	-40.56%	-48.59%	-52.83%	-55.50%	-60.99%	-36.51%	-92.36%
Increase Default Threshold to 90%	CVaR 99.97th Percentile - Dfltpnt 90%	0.8638%	0.4535%	0.4609%	0.4273%	0.2547%	0.2983%	0.2435%	0.1915%
	Percent Change from Base	22.43%	214.33%	28.46%	60.74%	21.46%	128.73%	268.11%	520.77%
Decrease Default Threshold to 75%	CVaR 99.97th Percentile - Dfltpnt 75%	0.0781%	0.0256%	0.0272%	0.0263%	0.0260%	0.0195%	0.0117%	0.0070%
	Percent Change from Base	-88.93%	-95.83%	-92.41%	-90.11%	-87.62%	-85.07%	-82.34%	-77.29%



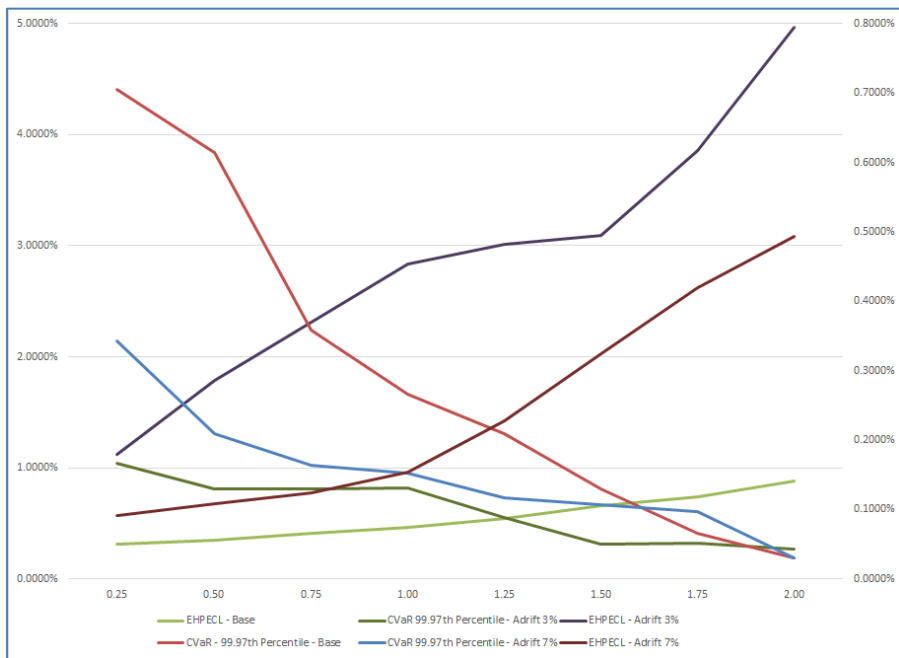
risk measures and the CEV parameter  $\theta$ , in that we as we increase (decrease)  $\mu$  the rate of decrease of the CVaR-99.97 in  $\theta$  is lower (higher) as compared to the base. For example, for an increase (decrease) in  $\mu$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in CVaR-99.97 are about  $[-71\%, -67\%]$  ( $[91\%, 210\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in range in the CVaR-99.97 are about  $[-69\%, -1\%]$  ( $[46\%, 760\%]$ ). On the other hand, when we increase or decrease  $\mu$ , we also observe an asymmetry in the EHPECL, as in both cases it is increasing in the CEV parameter  $\theta$ , albeit at a lower rate (at about the same rate) shocking in an increasing (decreasing) direction. For example, for an increase (decrease) in  $\mu$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in EHPECL are about  $[-71\%, -67\%]$  ( $[250\%, 500\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in the EHPECL are in about the ranges of  $[-57\%, -43\%]$  ( $[370\%, 450\%]$ ).

Third, let us consider the effect on the CVaR-99.97 and on the EHPECL of shocking LGD volatility  $\zeta_0$ , which is shown in the 5<sup>th</sup> and 6<sup>th</sup> panels of Table 4.3. We observe that when  $\zeta_0$  is increased (decreased) to 6% (2%), as expected the CVaR99.97 and EHPECL both increase (decrease) from the base case, which is intuitive in that greater (lesser) LGD volatility implies an increased (decreased) probability of default. Note an asymmetry that the risk measure CVaR99.97 (EHPECL) generally changes in about the same (greater) proportion to the increasing changes, whereas the CVaR99.97 and EHPECL both generally changes in about lesser proportion to the decreasing changes, in the LGD volatility parameter across values of the CEV parameter  $\theta$ , with the CVaR99.97 (EHPECL) increasing (decreasing) about on the order of 40% (100%) in the former and both declining about 20% in the latter. Furthermore, note that we observe some symmetries in the change in the relationship between the risk measures and the CEV parameter  $\theta$ , in that as we increase or decrease  $\zeta_0$  the rate of decrease of the CVaR-99.97 in  $\theta$  is about the same as compared to the base. For example, for an increase (decrease) in  $\zeta_0$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in CVaR-99.97 are about  $[19.5\%, 63.8\%]$  ( $[-45.2\%, -3.0\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in range in the CVaR-99.97 are about  $[23.45\%, 68.7\%]$  ( $[-52.9\%, -1.2\%]$ ). Furthermore, when we increase or decrease  $\zeta_0$ , unlike the CVaR997 we observe an symmetry in the EHPECL, as in the case of increasing (decreasing)  $\zeta_0$  it is increasing in the CEV parameter  $\theta$  at about the same (at a faster rate). For example, for an increase (decrease) in  $\zeta_0$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in EHPECL are about  $[12.5\%, 33.1\%]$  ( $[-19.3\%, -6.2\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes are in the EHPECL are about in the ranges  $[18.7\%, 28.1\%]$  ( $[-46.8\%, -26.0\%]$ ).

**Figure 4.11: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) for Asset Price Volatility – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 8\}$ )**



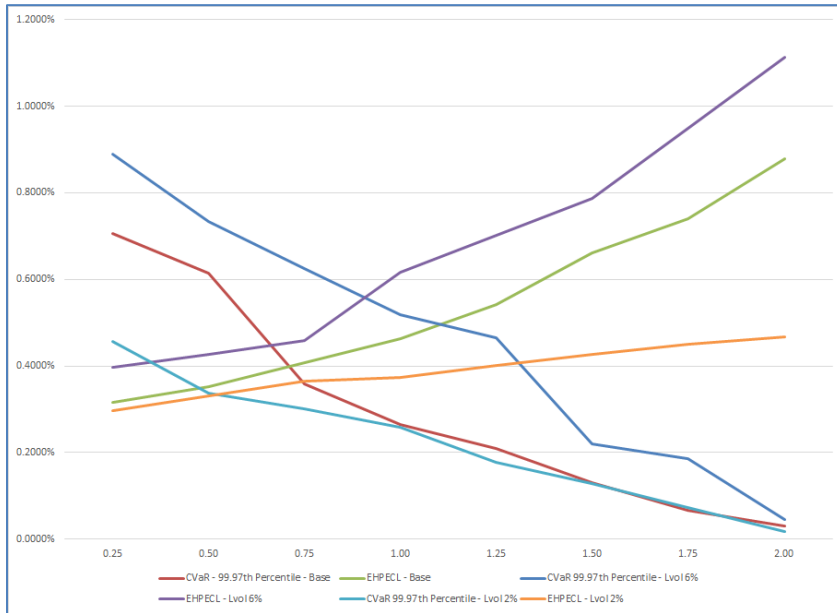
**Figure 4.12: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) for Asset Price Drift – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 8\}$ )**



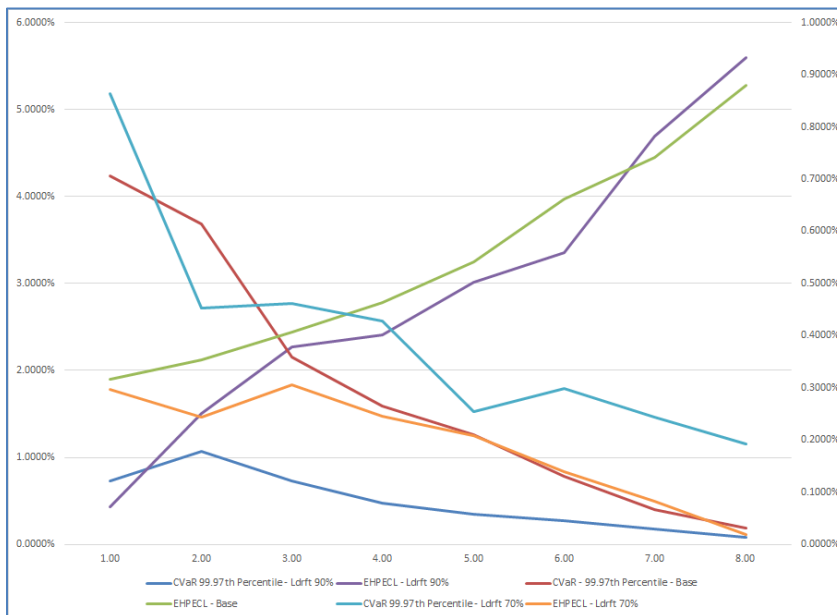
Fourth, let us consider the effect on the CVaR-99.97 and on the EHPECL of shocking the LGD drift  $\nu_0$ , which is shown in the 7<sup>th</sup> and 8<sup>th</sup> panels of Table 4.3. We observe that when  $\nu_0$  is increased (decreased) to 90% (70%), as expected the CVaR99.97 and EHPECL both increase (decrease) from the base case, which is intuitive in that greater (lesser) LGD drift implies an increased (decreased) probability of default. Note a symmetry that both the risk measures CVaR99.97 and EHPECL generally changes in a greater proportion to the changes in the LGD drift parameter across values of the CEV parameter  $\theta$ , with the CVaR99.97 (EHPECL) increasing (decreasing) about on the order of over 100% (50%) as compared to a shift in  $\nu_0$  of about 10% in either direction. Furthermore, note that we observe further symmetries in the change in the relationship between the risk measures and the CEV parameter  $\theta$ , in that as we increase or decrease  $\nu_0$  the rate of change in either the CVaR-99.97 or the LROP in  $\theta$  is greater as compared to the base. For example, for an increase (decrease) in  $\nu_0$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in CVaR-99.97 are about  $[3.9\%, 104.1\%]$  ( $[-52.8\%, -1.3\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in range in the CVaR-99.97 are about  $[65.4\%, 165.8\%]$  ( $[-92.4\%, -36.5\%]$ ). Furthermore, when we increase or decrease  $\nu_0$ , like the CVaR99.97 we observe a symmetry in the EHPECL, as in the case of increasing or decreasing  $\nu_0$  it is increasing in the CEV parameter  $\theta$  at a faster rate in either case. For example, for an increase (decrease) in  $\nu_0$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in EHPECL are about  $[35.0\%, 457.7\%]$  ( $[-47.0\%, -6.1\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes are in the EHPECL are about in the ranges  $[407.1\%, 536.3\%]$  ( $[-98.0\%, -61.7\%]$ ).

Finally, let us consider the effect on the CVaR-99.97 and on the EHPECL of shocking the default threshold  $c$ , which is shown in the bottom two panels of Table 4.3. We observe that when  $c$  is increased (decreased) to 90% (75%), as expected the CVaR99.97 and EHPECL both increase (decrease) from the base case, which is intuitive in that greater (lesser)  $c$  implies an increased (decreased) probability of default. Note a symmetry that both the risk measures CVaR99.97 and EHPECL) generally changes in a greater proportion to the changes  $c$  across values of the CEV parameter  $\theta$ , with the CVaR99.97 (EHPECL) increasing (decreasing) about on the order of over 100% to 1,500% (50%) as compared to a shift in  $c$  of about 10% (5%) in an increasing (decreasing) direction. Furthermore, note that we observe a symmetry in the change in the relationship between the risk measures and the CEV parameter  $\theta$ , in that as we increase or decrease  $c$  the rate of change in the CVaR-99.97 or EHPECL in  $\theta$  is greater in either case as compared to the base. For example, for an increase (decrease) in  $c$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in CVaR-99.97 are about  $[22.4\%, 214.3.1\%]$  ( $[-95.8\%, -88.9\%]$ ), whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes in range in the CVaR-99.97 are about  $[21.5\%, 520.8\%]$  ( $[-87.6\%, -77.3\%]$ ). For example, for an increase (decrease) in  $c$ , in the non-bubble region  $\theta \in [0.25, 1.00]$  the percent changes range in EHPECL are about

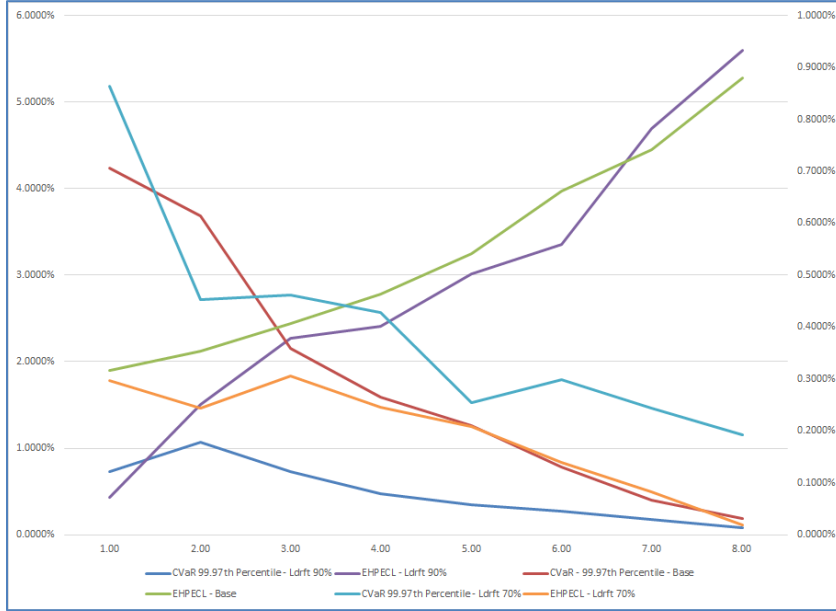
**Figure 4.13: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) for Loss-Given-Default Volatility – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 8\}$ )**



**Figure 4.14: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) for Loss-Given-Default Drift – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 8\}$ )**



**Figure 4.15: Sensitivity Analysis of 99.97<sup>th</sup> Percentile Value-at-Risk (VaR) and Expected Holding Period Credit Loss (EHPCL) for default Point – Stochastic Simulation of CEV Asset Value Process for Various Values of the CEV Parameter ( $\theta = \{0.25 + i \times 0.25 \text{ for } i = 0, 1, \dots, 8\}$ )**



$[406.2\%, 1178.8\%]$   $([-85.9\%, -41.8\%])$ , whereas in bubble region  $\theta \in [1.25, 2.00]$  the percent changes are in the EHPECL are about in the ranges  $[1476.2\%, 3506.2\%]$   $([-157.7\%, -107.1\%])$ .

## 5 Conclusion and Future Directions

In this study we have analyzed the impact of asset price bubbles on credit risk measures for a representative firm subject to default risk. We have constructed various hypothetical economies, both having and also not having asset price bubbles, in a structural credit model framework. We have simulated a firm's value process in each of these economies, computing the firm's standard risk measures, in continuous time, performing a simulation experiment of a 2-dimensional system of SDEs for asset value determining PD and a correlated a LGD process. Comparing bubble to non-bubble economies, it has been shown that asset price bubbles may cause an obligor's traditional credit risk measures (such as EL or CVaR) to decline, due to a reduced standard deviation and a reduced right skewness of the credit loss distribution (driven by an augmented right skewness of the asset value distribution). We have developed the new EHPCL credit risk measure to account for losses associated with a bubble bursting, which behaves intuitively as we transition from non-bubble to bubble economies.

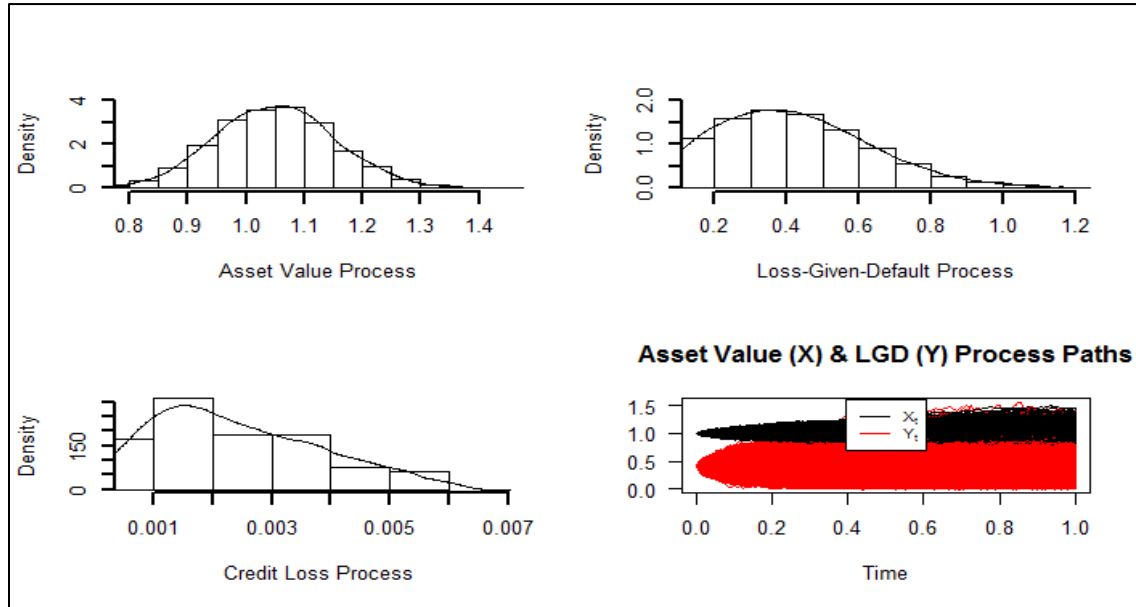
The results of our experiment have demonstrated that the existence of an asset price bubble, which occurs for certain parameter settings in the CEV model, results in the firm asset value distribution to have both a lower standard deviation and a greater right skewness. We have demonstrated that this augmented right skewness, in conjunction with a reduced variance of a firm's return due to bubble expansion, results in a reduction of the right skewness in the distribution of the default rate and a lower PD, which in combination with a lower mean of the LGD process, results in a credit loss distribution having both lower mean right skewness and lower standard deviation. This in turn implies causes the firm's CVaR measures to decline, an understatement in the credit risk of the firm. Based on these measures alone, their declining values imply that in the presence of asset price bubbles, less credit capital is required. However, we have shown that according to the new EHPCL, this conclusion is incorrect, as this EHPCL credit loss measure increases in bubble economies and is due to bubble bursting, which causes significant firm value losses, and therefore credit losses, on the bubble-bursting paths.

In the detailed analysis of the results we have shown that the various credit loss metrics are all declining functions of the CEV parameter: the ECL is decreasing monotonically in  $\theta$ , and also all of the CVaR measures decline monotonically and at an accelerating pace in  $\theta$ , with the steepness of the declining actually becoming accentuated as the confidence level becomes more conservative. This was shown to be reflected in the declining skewness of the credit loss distribution in  $\theta$ , implying that skewness this is a robust indicator of a bubble, but however does not have the benefit of having an intuitive economic interpretation as does the EHPCL. We also perform a sensitivity analysis of the SDE parameters upon the resulting credit risk measures, as well as the changes in their relationship to the CEV parameter, illustrating an application of an important model validation procedure. Results show that while the CVaR and EHPCL react intuitively in direction with respect to changes in these parameters, the magnitude of the shift relative to that of the parameters, as well as the change in the relationship between these measures and the CEV parameter, vary across parameter..

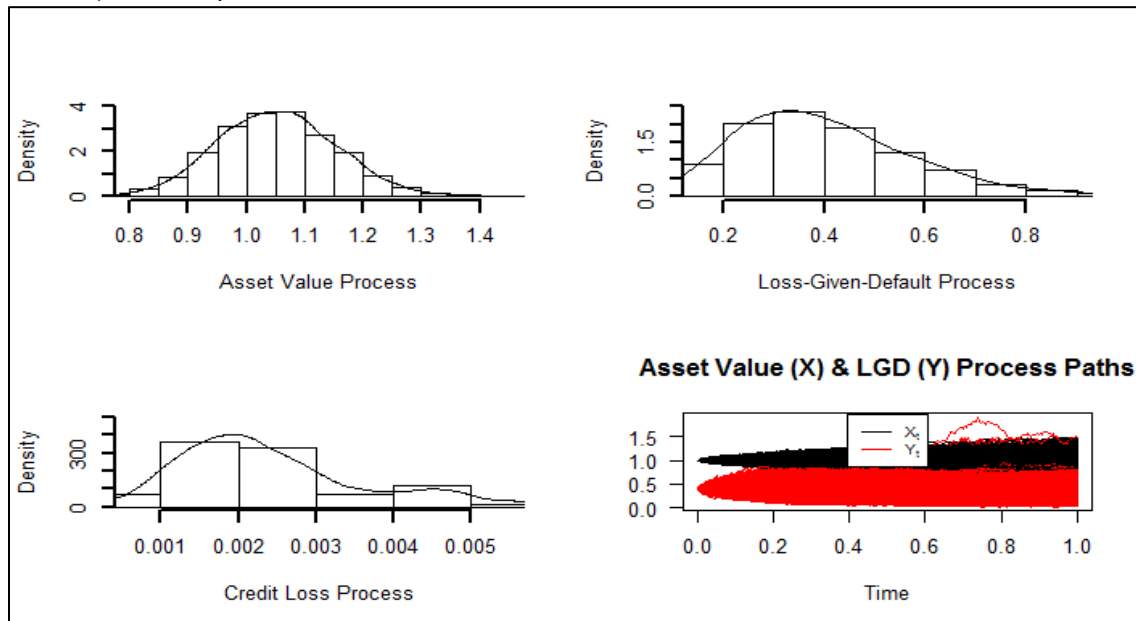
There are various avenues down which we may proceed in the interest of pursuing additional research. One such direction would be an extension to a structural model that can admit differential seniority for LGD, an option-theoretic approach along the lines of Jacobs (2011, 2012). Another potential sequel to this study would be to analysis of a real portfolio of equities and empirical calibration, along the lines of Jarrow et al (2015), although in the context of credit as opposed to market risk. Finally, we may investigate more general stochastic diffusion models for asset value, such as the incorporation of jump processes.

## 6 Appendix: Stochastic Simulation Histograms and Paths

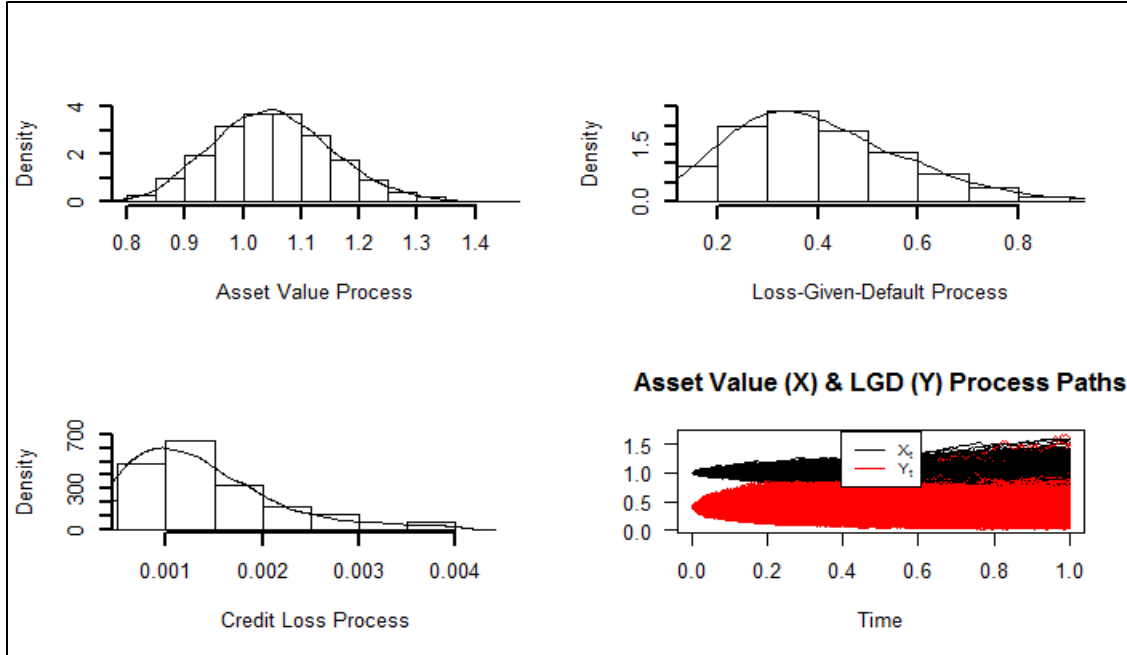
**Figure 6.1: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 0.25$ )**



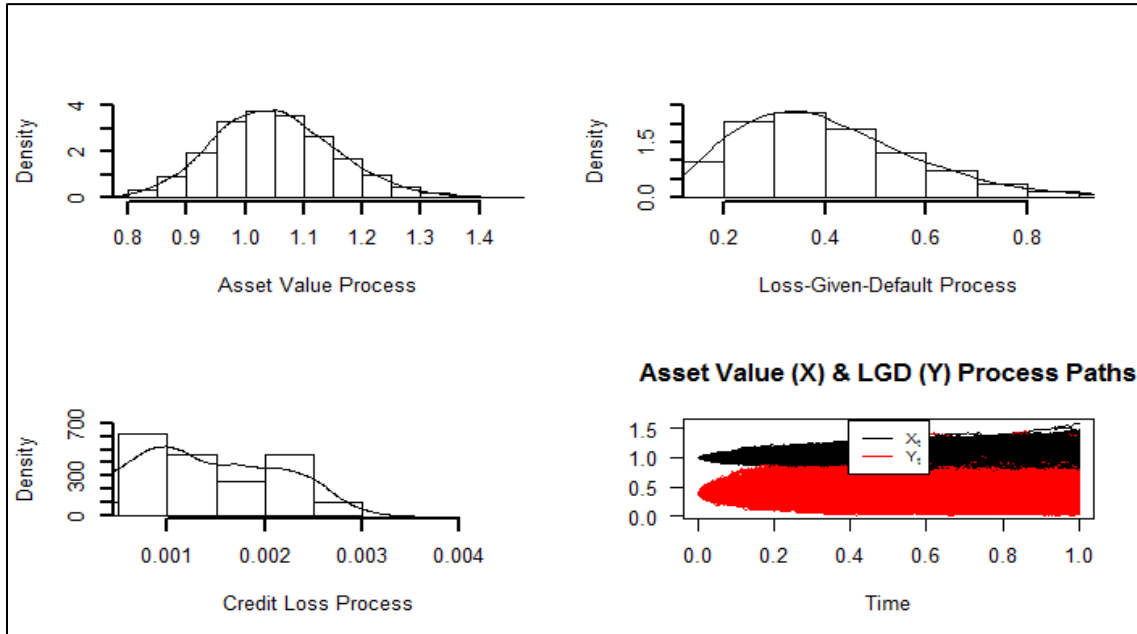
**Figure 6.2: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 0.50$ )**



**Figure 6.3: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 0.75$ )**

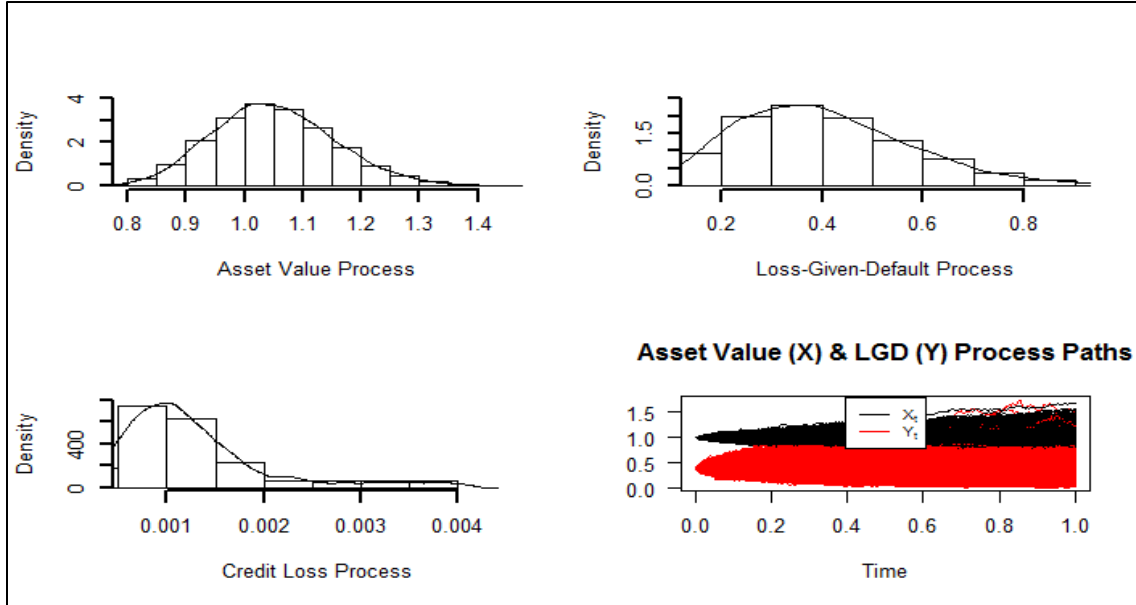


**Figure 6.4: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 1.00$ )**

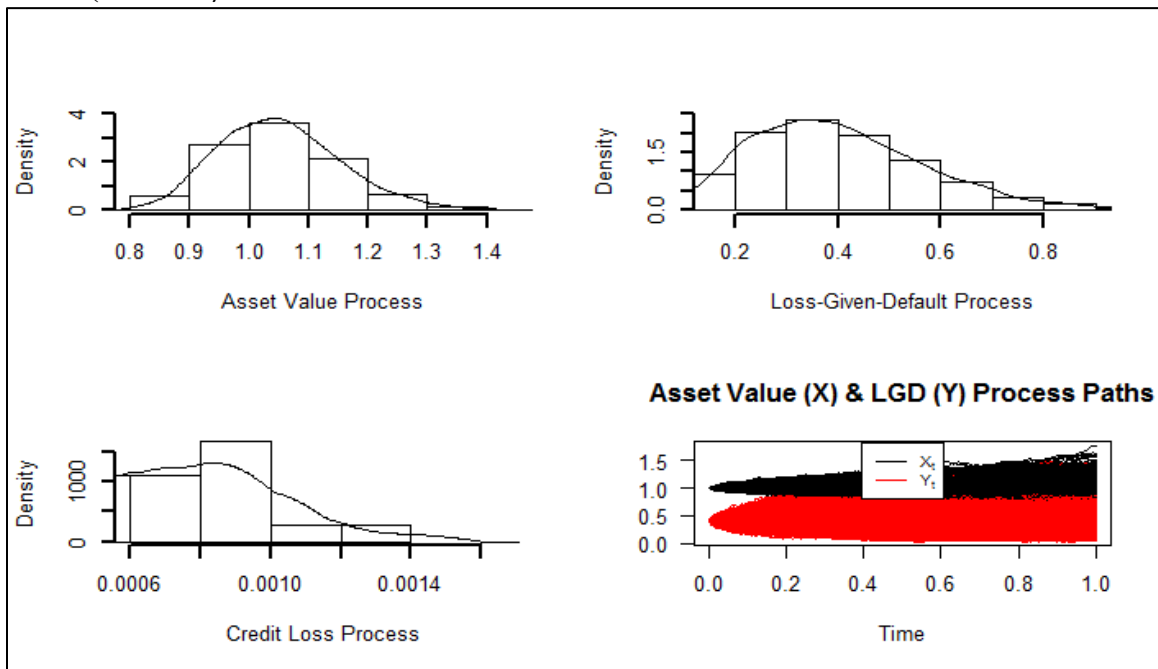




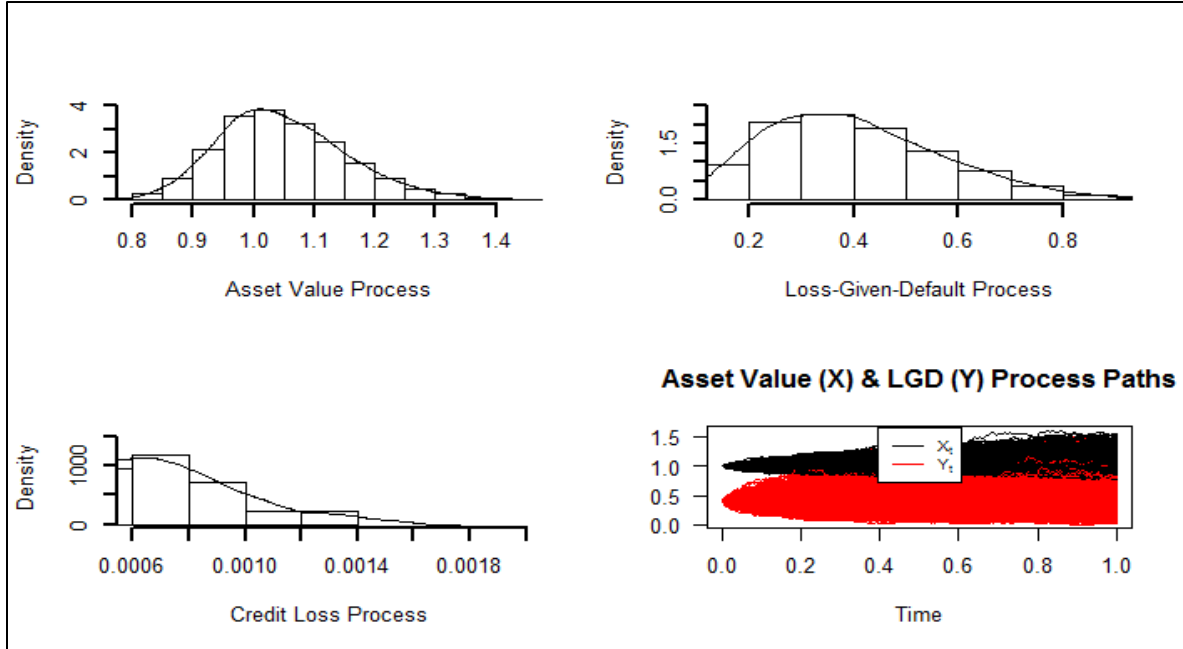
**Figure 6.5: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 1.25$ )**



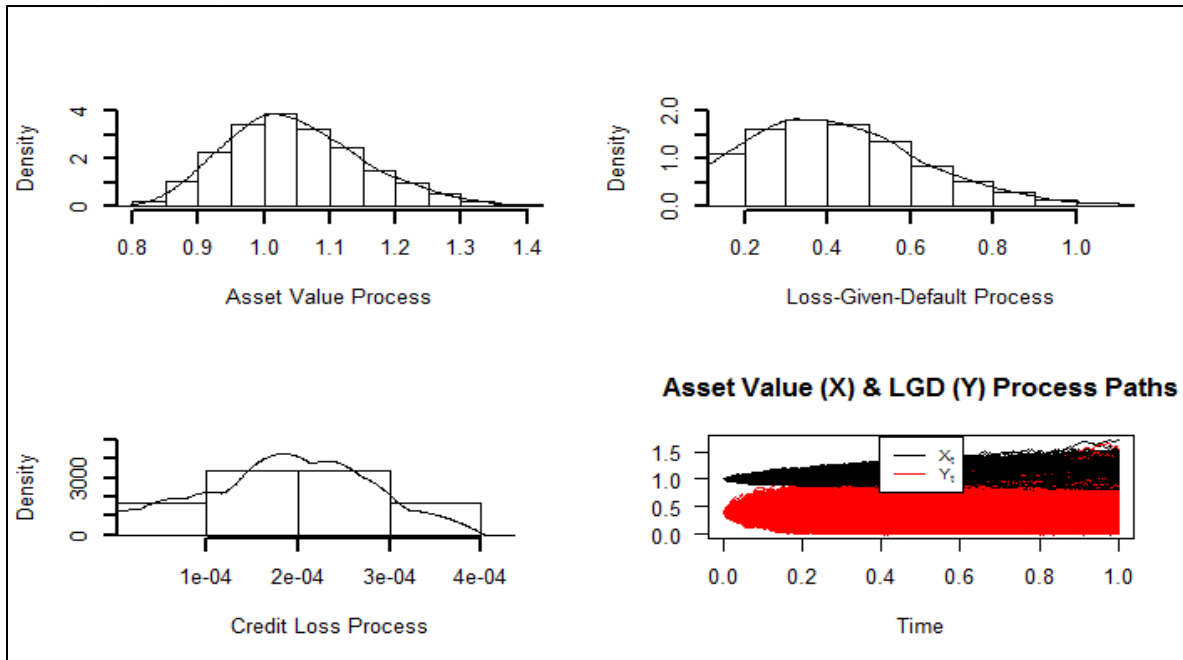
**Figure 6.6: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \varsigma_0 = 0.4, \varsigma_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 1.50$ )**



**Figure 6.7: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 1.75$ )**



**Figure 6.8: Stochastic Simulation of CEV Asset Value Process ( $V_0 = 1, \mu = 0.05, D = 0.80$   $\sigma = 0.10$ ), Correlated CEV Mean-Reverting Loss-Given-Default Process ( $\nu_0 = 0.80$ ,  $\nu_1 = 0.40, L_0 = 0.40, \zeta_0 = 0.4, \zeta_1 = 0.25$ ) and Credit Loss Processes – Histograms and Sample Paths ( $\theta = 2.00$ )**



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