An option theoretic model for ultimate loss-given-default with systematic recovery risk and stochastic returns on defaulted debt

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1. Introduction and Summary

Loss-given-default (LGD),[2] the loss severity on defaulted obligations, is a critical component of risk management, pricing and portfolio models of credit. This is among the three primary determinants of credit risk, the other two being the *probability of default* (PD) and *exposure of default* (EAD). However, LGD has not been as extensively studied, and is considered a much more daunting modeling challenge than other components, such as PD. Starting with the seminal work by Altman (1968), and after many years of actuarial tabulation by rating agencies, predictive modeling of PD is currently in a mature stage. The focus on PD is understandable, as traditionally credit models have focused on systematic components of credit risk which attract risk premia, and unlike PD, determinants of LGD have been ascribed to idiosyncratic borrower specific factors. However, now there is an ongoing debate about whether the risk premium on defaulted debt should reflect systematic risk, in particular whether the intuition that LGDs should rise in worse states of the world is correct, and how this could be refuted empirically given limited and noisy data (Carey and Gordy, 2007).

The recent heightened focus on LGD is evidenced by the flurry of research into this relatively neglected area (Acharya et al [2007], Carey and Gordy [2007], Altman et al [2001, 2003, 2005], Altman [2006], Gupton et al [2000, 2005], Araten et al [2004], Frye [2000 a,b,c, 2003], Jarrow [2001]). This has been motivated by the large number of defaults and near simultaneous decline in recovery values observed at the trough of the last credit cycle circa 2000-2002, regulatory developments such as Basel II (BIS [2003, 2005, 2006], OCC et al [2007]) and the growth in credit markets. However, obstacles to better understanding and predicting LGD, including dearth of data and the lack of a coherent theoretical underpinning, have continued to challenge researchers. In this paper, we hope to contribute to this effort by synthesizing advances in financial theory to build a model of LGD that is consistent with a priori expectations and stylized facts, internally consistent and amenable to rigorous validation. In addition to answering the many questions that academics have, we further aim to provide a practical tool for risk managers, traders and regulators in the field of credit.

LGD may be defined variously depending upon the institutional setting or modeling context, or the type of instrument (traded bonds vs. bank loans) versus the credit risk model (pricing debt instruments subject to the risk of default vs. expected losses or credit risk capital). In the

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2 This is equivalent to one minus the *recovery rate*, or dollar recovery as a proportion of par, or EAD assuming all debt becomes due at default. We will speak in terms of LGD as opposed to recoveries with a view toward credit risk management applications.

case of bonds, one may look at the price of traded debt at either the initial credit event,[3] the market values of instruments received at the resolution of distress[4] (Keisman et al, 2000; Altman et al, 1996) or the actual cash-flows incurred during a workout.[5] When looking at loans that may not be traded, the eventual loss per dollar of outstanding balance at default is relevant (Asarnow et al, 1995; Araten et al, 2004). There are two ways to measure the latter – the *accounting LGD* refers to nominal loss per dollar outstanding at default,[6] while the *economic LGD* refers to the discounted cash flows to the time of default taking into consideration when cash was received. The former is used in setting reserves or a loan loss allowance, while the latter is an input into a regulatory or economic credit capital model.

In this study we develop various theoretical models for ultimate loss-given-default in the Merton (1974) structural credit risk model framework. We consider an extension that allows for differential seniority within the capital structure, an independent recovery rate process, representing undiversifiable recovery risk, with stochastic drift. The comparative statics of this model are analyzed and compared to a baseline model, all of these in a framework that incorporates an optimal foreclosure threshold (Carey and Gordy, 2007). In the empirical exercise, we calibrate alternative models for ultimate LGD on bonds and loans having both trading prices at default and at resolution of default, utilizing an extensive sample of rated defaulted firms in the period 1987-2008 (Moody's Ultimate Recovery Database™ - URD™), 800 defaults (bankruptcies and out-of-court settlements of distress) that are largely representative of the US large corporate loss experience, for which we have the complete capital structures and can track the recoveries on all instruments to the time of default to the time of resolution.

We find that parameter estimates vary significantly across models and recovery segments. estimated volatilities of the recovery rate processes, as well as of their random drifts, are found to increasing in seniority, in particular for bank loans as compared to bonds. We interpret this as reflecting greater risk in the ultimate recovery for higher ranked instruments having lower expected loss severities (or ELGDs). Analyzing the implications of our model for the quantification of downturn LGD, we find the later to be *declining* in expected LGD, higher for worse ranked instruments, increasing in the correlation between the process driving firm default and recovery on collateral, and increasing in the volatility of the systematic factor specific to the recovery rate process or the volatility of the drift in such. Finally, we validate the leading model derived herein in an out-of-time and out-of-sample bootstrap exercise, comparing it to a high-dimensional regression model, and to a non-parametric benchmark based upon the same data, where we find our model to compare favorably. We conclude that our model is worthy of consideration to risk managers, as well as supervisors concerned with advanced IRB under the Basel II capital accord.

This paper is organized as follows. Section 2 reviews the literature, focusing on the treatment of LGD in theoretical credit models, both academic and practitioner. Section 3 presents the theoretical framework. Section 4 discusses comparative statics of the alternative models. Section 5 describes the econometric framework. Section 6 describes the data used in this study and presents the calibration analysis of structural model parameters. In Section 7 we

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[3] By default we mean either bankruptcy (Chapter 11) or other financial distress (payment default). In a banking context, this defined as synonymous with respect to non-accrual on a discretionary or non-discretionary basis. This is akin to the notion of default in Basel, but only proximate.

[4] Note that this may be either the value of pre-petition instruments received valued at emergence from bankruptcy, or the market values of new securities received in settlement of a bankruptcy proceeding or as the result of a distressed restructuring.

[5] Note that the former may viewed as a proxy to this, the pure economic notion.

[6] In the context of bank loans, this is the cumulative net charge-off as a percent of book balance at default (the *net charge-off rate)*.

discuss the implications of our modeling framework for downturn LGD. In Section 8 we perform an out-of-sample validation of our model and two alternative benchmarks. Finally, Section 9 concludes and discusses directions for future research.

2. Review of the literature

In this section we will examine the way in which different types of theoretical credit risk models have treated LGD – assumptions, implications for estimation and application. Credit risk modeling was revolutionized by the approach of Merton (1974), who built a theoretical model in the option pricing paradigm of Black and Scholes (1973), which has come known to be the *structural approach*. Equity is modeled as a call option on the value of the firm, with the face value of zero coupon debt serving as the strike price, which is equivalent to shareholders buying a put option on the firm from creditors with this strike price. Given this capital structure, log-normal dynamics of the firm value and the absence of arbitrage, closed form solutions for the default probability and the spread on debt subject to default risk can be derived. The LGD can be shown to depend upon the parameters of the firm value process as is the PD, and moreover is directly related to the latter, in that the expected residual value to claimants is increasing (decreasing) in firm value (asset volatility or the level of indebtedness). Therefore, LGD is not independently modeled in this framework; this was addressed in much more recent versions of the structural framework (Frye [2000 a,b], Dev and Pykhtin [2002], Pykhtin [2003]).

Extensions of Merton (1974) relaxed many of the simplifying assumptions of the initial structural approach. Complexity to the capital structure was added by Black and Cox (1976) and Geske (1977), with subordinated and interest-paying debt, respectively. The distinction between long- and short-term liabilities in Vasicek (1984) was the precursor to the KMV™ model. However, these models had limited practical applicability, the standard example being evidence of Jones, Mason and Rosenfeld (1984) that these models were unable to price investment-grade debt any better than a naïve model with no default risk. Further, empirical evidence in Franks and Touros (1989) showed that the adherence to absolute priority rules (APR) assumed by these models are often violated in practice, which implies that the mechanical negative relationship between expected asset value and LGD may not hold. Longstaff & Schwartz (1995) incorporate into this framework a stochastic term structure with a PD-interest rate correlation. Other extensions include Kim at al (1993) and Hull & White (2002), who examine the effect of coupons and the influence of options markets, respectively.

Partly in response to this, a series of extensions ensued, the so-called “second generation” of structural form credit risk models (Altman, 2003). The distinguishing characteristic of this class of models is the relaxation of the assumption that default can only occur at the maturity of debt – now default occurs at any point between debt issuance and maturity when the firm value process hits a threshold level. The implication is that LGD is exogenous relative to the asset value process, defined by a fixed (or exogenous stochastic) fraction of outstanding debt value. This approach can be traced to the barrier option framework as applied to risky debt of Black and Cox (1976).

All structural models suffer from several common deficiencies. First, reliance upon an unobservable asset value process makes calibration to market prices problematic, inviting model risk. Second, the limitation of assuming a continuous diffusion for the state process implies that the time of default is perfectly predictable (Duffie and Lando, 2001). Finally, the inability to model spread or downgrade risk distorts the measurement of credit risk. This gave rise to the *reduced form approach* to credit risk modeling (Duffie and Singleton, 1999), which instead of conditioning on the dynamics of the firm, posit exogenous stochastic processes for PD and LGD. These models include (to name a few) Litterman & Iben (1991), Madan & Unal

(1995), Jarrow & Turnbull (1995), Lando (1998) and Duffie (1998). The primitives determining the price of credit risk are the term structure of interest rates (or short rate), and a default intensity and an LGD process. The latter may be correlated with PD, but it is exogenously specified, with the link of either of these to the asset value (or latent state process) not formally specified. However, the available empirical evidence (Duffie and Singleton, 1999) has revealed these models deficient in generating realistic term structures of credit spreads for investment and speculative grade bonds simultaneously. A hybrid reduced – structural form approach of Zhou (2001), which models firm value as a jump diffusion process, has had more empirical success, especially in generating a realistic negative relationship between LGD and PD (Altman et al, 2006).

The fundamental difference between reduced and structural form models is the unpredictability of defaults: PD is non-zero over any finite time interval, and the default intensity is typically a jump process (eg Poisson), so that default cannot be foretold given information available the instant prior. However, these models can differ in how LGD is treated. The *recovery of treasury* assumption of Jarrow & Turnbull (1995) assumes that an exogenous fraction of an otherwise equivalent default-free bond is recovered at default. Duffie and Singleton (1999) introduce the *recovery of market value* assumption, which replaces the default-free bond by a defaultable bond of identical characteristics to the bond that defaulted, so that LGD is a stochastically varying fraction of market value of such bond the instant before default. This model yields closed form expressions for defaultable bond prices and can accommodate the correlation between PD and LGD; in particular, these stochastic parameters can be made to depend on common systematic or firm specific factors. Finally, the *recovery of face value* assumption (Duffie [1998], Jarrow et al [1997]) assumes that LGD is a fixed (or seniority specific) fraction of par, which allows the use of rating agency estimates of LGD and transition matrices to price risky bonds.

It is worth mentioning the treatment of LGD in credit models that attempt to quantify unexpected losses analogously to the *value-at-risk* (VaR) market risk models, so-called *credit VaR* models (Creditmetrics™ [Gupton et al, 1997], KMV CreditPortfolioManager™ [Vasicek, 1984], CreditRisk$^+$™ [Credit Suisse Financial Products, 1997], CreditPortfolioView™ [Wilson, 1998]). These models are widely employed by financial institutions to determine expected credit losses as well as economic capital (or unexpected losses) on credit portfolios. The main output of these models is a probability distribution function for future credit losses over some given horizon, typically generated by simulation of analytical approximations, as it is modeled as highly non-normal (asymmetrical and fat-tailed). Characteristics of the credit portfolio serving as inputs are LGDs, PDs, EADs, default correlations and rating transition probabilities. Such models can incorporate credit migrations (*mark-to-market mode* - MTM), or consider the binary default vs. survival scenario (*default mode* - DM), the principal difference being that in addition an estimated transition matrix needs to be supplied in the former case. Similarly to the reduced form models of single name default, LGD is exogenous, but potentially stochastic. While the marketed vendor models may treat LGD as stochastic (eg a draw from a beta distribution that is parameterized by expected moments of LGD), there are some more elaborate proprietary models that can allow LGD to be correlated with PD.

We conclude our discussion of theoretical credit risk models and the treatment of LGD by considering recent approaches, which are capable of capturing more realistic dynamics, sometimes called “hybrid models”. These include Frye (2000a, 2000b), Jarrow (2001), Bakshi et al (2001), Jarrow et al (2003), Pykhtin (2003) and Carey & Gordy (2007). Such models are motivated by the conditional approach to credit risk modeling, credited to Finger (1999) and Gordy (2000), in which a single systematic factor derives defaults. In this more general setting, they share in common the feature that dependence upon a set of systematic factors can induce an endogenous correlation between PD & LGD. In the model of Frye (2000a, 2000b), the mechanism that induces this dependence is the influence of systematic factors upon the value of loan collateral, leading to a lower recoveries (and higher loss severity) in periods where default rates rise (since asset values of obligors also depend upon the same factors). In a reduced form setting, Jarrow (2001) introduced a model of co-dependent LGD and PD implicit in debt and equity prices.[7]

3. Theoretical model

The model that we propose is an extension of Black and Cox (1976). The baseline mode features perpetual corporate debt, a continuous and a positive foreclosure boundary. The former assumption removes the time dependence of the value of debt, thereby simplifying the solution and comparative statics. The latter assumption allows us to study the endogenous determination of the foreclosure boundary by the bank, as in Carey and Gordy (2007). We extend the latter model by allowing the coupon on the loan to follow a stochastic process, accounting for the effect of illiquidity. Note that in this framework, we assume no restriction on asset sales, so that we do not consider strategic bankruptcy, as in Leland (1994) and Leland and Toft (1996).

Let us assume a firm financed by equity and debt, normalized such that the total value of perpetual debt is 1, divided such that there is a single loan with face value λ and a single class of bonds with a face value of $1-\lambda$. The loan is senior to that bond, and potentially has covenants which permit foreclosure. The loan is entitled to a continuous coupon at a rate c, which in the baseline model we take as a constant, but may evolve randomly. Equity receives a continuous dividend, having a constant and a variable component, which we denote as $\delta+\rho V_t$, where V_t is the value of the firm's assets at time t. We impose the restriction that $0 \leq \rho \leq r \leq c$, where r is the constant risk-free rate. The asset value of the firm, net of coupons and dividends, follows a geometric Brownian motion with constant volatility σ:

$$\frac{dV_t}{V_t} = \left(r - \rho - \frac{C}{V_t}\right)dt + \sigma dZ_t \tag{3.1}$$

Where in (3.1) we denote the fixed cash outflows per unit time as:

$$C = c\lambda + \gamma\left(1-\lambda\right) + \delta \tag{3.2}$$

Where in (3.2), γ and δ are the continuous coupon rate on the bond and dividend yield on equity, respectively. Default occurs at time t and is resolved after a fixed interval τ, at which point dividend payments cease, but the loan coupon continues to accrue through the settlement period. At the point of emergence, loan holders receive $\left(\lambda \exp\left(c\tau\right), V_{t+\tau}\right)^{-}$, or the minimum of the legal claim or the value of the firm at emergence. We can value the loan at resolution, under risk neutral measure, using the standard Merton (1974) formula. Denote the total legal claim at default by:

$$D = \lambda \exp\left(\tau c\right) + \left(1-\lambda\right). \tag{3.3}$$

This follows from the assumption that the coupon c on the loan with face value λ continues to accrue at the contractual rate throughout the resolution period τ, whereas the bond with face value $1-\lambda$ does not.

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7 Jarrow (2001) also has the advantage of isolating the liquidity premium embedded in defaultable bond spreads.

Thus far we have assumed that the senior bank creditors foreclose on the bank when the value of assets is V_t, where t is the time of default. However, this is not realistic, as firm value fluctuates throughout the bankruptcy or workout period, and we can think that there will be some *foreclosure boundary* (denoted κ) below which foreclosure is effectuated. Furthermore, in most cases there exists a *covenant boundary*, above which foreclosure cannot occur, but below which it may occur as the borrower is in violation of a contractual provision. For the time being, let us ignore the latter complication, and focus on the optimal choice of κ by the bank. In the general case of time dependency in the loan valuation equation $F\left(V_t \mid \lambda, \sigma, r, \tau\right)$, following Black and Cox (1976), we have to solve a second-order partial differential equation. Following Carey and Gordy (2007), we modify this such that the value of the loan at the threshold is not a constant, but simply equal to the recovery value of the loan at the default time. Second, we remove the time dependency in the value of the perpetual debt. It is shown in Carey and Gordy (2007) that under these assumptions, so long as there are positive and fixed cash flows to claimants other than the bank, $\gamma\left(1-\lambda\right)>0$ or $\delta > 0$, then there exists a finite and positive solution κ^*, the optimal foreclosure boundary.

We model undiversifiable recovery risk by introducing a separate process for recovery on debt, R_t. This can be interpreted as the state of collateral underlying the loan or bond. R_t is a geometric Brownian process that depends upon the Brownian motion that drives the return on the firm's assets Z_t, an independent Brownian motion W_t and a random instantaneous mean α_t:

$$\frac{dR_t}{R_t} = \alpha_t dt + \beta dZ_t + \upsilon dW_t \tag{3.6}$$

$$d\alpha_t = \kappa_\alpha\left(\bar{\alpha} - \alpha_t\right)dt + \eta dB_t \tag{3.7}$$

Where the volatility parameter β represents the sensitivity of recovery to the source of uncertainty driving asset returns (or the "systematic factor"), implying that the instantaneous correlation between asset returns and recovery is given by $\frac{1}{dt}Corr_t\left(\frac{dA_t}{A_t} \times \frac{dR_t}{R_t}\right) = \sqrt{\beta\sigma}$. On the other hand, the volatility parameter υ represents the sensitivity of recovery to a source of uncertainty that is particular to the return on collateral, also considered a "systematic factor", but independent of the asset return process. The third source of recovery uncertainty is given by (3.7), where we model the instantaneous drift on the recovery by an Orhnstein-Uhlenbeck mean-reverting process, with κ_α the speed of mean-reversion, $\bar{\alpha}$ the long-run mean, η the constant diffusion term, and B_t is a standard Weiner process having instantaneous correlation with the source of randomness in the recovery process, given heuristically by $\varsigma = \frac{1}{dt}Corr_t\left(dB_t, dW_t\right)$. The motivation behind this specification is the overwhelming evidence that the mean LGD is stochastic.

Economic LGD on the loan is given by following expectation under physical measure:

$$LGD_\lambda^P\left(R_t, \alpha_t \mid \lambda, c, \beta, \upsilon, \kappa_\alpha, \eta, \varsigma, \tau\right) =$$

$$= 1 - \frac{\exp\left(-c\tau\right)}{\lambda} E_t\left(\min\left[\lambda \exp\left(c\tau\right), R_t \exp\left(\left(\alpha_t - \frac{\beta^2 + \nu^2}{2}\right)\tau + \beta Z_{t+\tau} + \nu W_{t+\tau}\right)\right]\right)$$

$$=1-\frac{\exp\left(\left(\alpha_t-c\right)\tau\right)}{\lambda}B\left(R_t,\alpha_t\mid\lambda\exp\left(c\tau\right),\hat{\sigma}_\tau,\alpha_t,\tau\right) \tag{3.8}$$

Where the modified option theoretic function $B\left(\bullet\right)$ is given by:

$$B\left(R_t,\alpha_t\mid\lambda\exp\left(c\tau\right),\hat{\sigma}_\tau,\alpha_t,\tau\right)=R_t\Phi\left(-d_+^{'}\right)+\exp\left(\left(c-\alpha_t\right)\tau\right)\lambda\Phi\left(d_-^{'}\right) \tag{3.9}$$

having arguments to the Gaussian distribution function $\Phi\left(z\right)=\frac{1}{2\pi}\int_{u=-\infty}^{z}e^{-\frac{u^2}{2}}du$:

$$d_\pm^{'}=\frac{1}{\hat{\sigma}_\tau\sqrt{\tau}}\left(\log\left(\frac{R_t}{\lambda\exp\left(c\tau\right)}\right)+\tau\left(\alpha_t\pm\frac{1}{2}\hat{\sigma}_\tau^2\right)\right) \tag{3.10}$$

A well-known result (Bjerksund, 1991) is that the maturity-dependent volatility $\hat{\sigma}_\tau$ is given by:

$$\hat{\sigma}_\tau=\left(\beta^2+\upsilon^2-\frac{\eta}{\kappa_\alpha}\left(2\sqrt{\beta^2+\upsilon^2}\varsigma-\frac{\eta}{\kappa_\alpha}\right)\right)\tau+\frac{2\eta}{\kappa_\alpha^2}\left(\sqrt{\beta^2+\upsilon^2}\varsigma-\frac{\eta}{\kappa_\alpha}\right)\left(1-e^{-\kappa_\alpha\tau}\right)+\frac{1}{2}\left(\frac{\eta}{\kappa_\alpha}\right)^2\left(1-e^{-2\kappa_\alpha\tau}\right) \tag{3.11}$$

The recovery to the bondholders is the expectation of the minimum of the positive part of the difference in the recovery and face value of the loan $\left[R_{t+\tau}-\lambda\exp\left(c\tau\right)\right]^+$ and the face value of the bond B, which is structurally identical to a compound option valuation problem (Geske, 1977):

$$LGD_B^P\left(V_t,R_t,\alpha_t\mid\lambda,c,\gamma,\beta,\upsilon,\kappa_\alpha,\eta,\varsigma,\tau_\lambda\right)=$$

$$=1-\frac{\exp\left(-\gamma,\tau_\lambda\right)}{B}E_t\left(\min\left[B,\max\left[R_t\exp\left(\left(\alpha_t-\frac{\beta^2+\nu^2}{2}\right)\tau_\lambda+\beta Z_{t+\tau}+\nu W_{t+\tau}\right)-\lambda\exp\left(c,\tau_\lambda\right),0\right]\right]\right) \tag{3.12}$$

where $R_{t+\tau}=R_t\exp\left(\left(\alpha_t-\frac{\beta^2+\nu^2}{2}\right)\tau_\lambda+\beta Z_{t+\tau_\lambda}+\nu W_{t+,\tau_\lambda}\right)$ is the value of recovery on the collateral at the time of resolution. We can easily write down the closed-form solution for the LGD on the bond according to the well-known formula for a compound option, where here the “outer option” is a put, and the “inner option” is a call. Let R^* be the critical level of recovery such that the holder of the loan is just breaking even:

$$\lambda\exp\left(c\tau_\lambda\right)=1-LGD_\lambda^P\left(R^*,\alpha_t\mid\lambda,c,\beta,\upsilon,\kappa_\alpha,\eta,\varsigma,\tau_\lambda\right) \tag{3.13}$$

where τ_λ is the time-to-resolution for the loan, which we assume to be prior to that for the bond, $\tau_\lambda<\tau_B$. Then the solution is given by:

$$LGD_B^P\left(R_t,\alpha_t\mid\lambda,c,\gamma,\beta,\upsilon,\kappa_\alpha,\eta,\varsigma,\tau_\lambda,\tau_B\right)=1-\frac{\exp\left(-\gamma\tau_b\right)}{B}\mathrm{B}\left(R_t,\alpha_t\mid\lambda,c,\gamma,\beta,\upsilon,\kappa_\alpha,\bar{\alpha},\eta,\varsigma,\tau_B\right) \tag{3.14}$$

$$\mathrm{B}\left(R_t,\alpha_t\mid\lambda,c,\gamma,\beta,\upsilon,\kappa_\alpha,\eta,\varsigma,\tau_\lambda,\tau_B\right)=$$

$$= B\exp(\alpha_t - \gamma)\Phi_2\left(-a_-, b_-; -\sqrt{\frac{\tau_\lambda}{\tau_B}}\right) - R_t\Phi_2\left(-a_+, b_+; -\sqrt{\frac{\tau_\lambda}{\tau_B}}\right) + \lambda\exp(c\tau_\lambda)\Phi(-a_-) \tag{3.15}$$

$$a_\pm = \frac{1}{\hat{\sigma}_\tau\sqrt{\tau_\lambda}}\left(\log\left(\frac{R_t}{R^*}\right) + \tau_\lambda\left(\alpha_t \pm \frac{1}{2}\hat{\sigma}_\tau^2\right)\right) \tag{3.16}$$

$$b_\pm = \frac{1}{\hat{\sigma}_\tau\sqrt{\tau_B}}\left(\log\left(\frac{R_t}{B}\right) + \tau_B\left(\alpha_t \pm \frac{1}{2}\hat{\sigma}_\tau^2\right)\right) \tag{3.17}$$

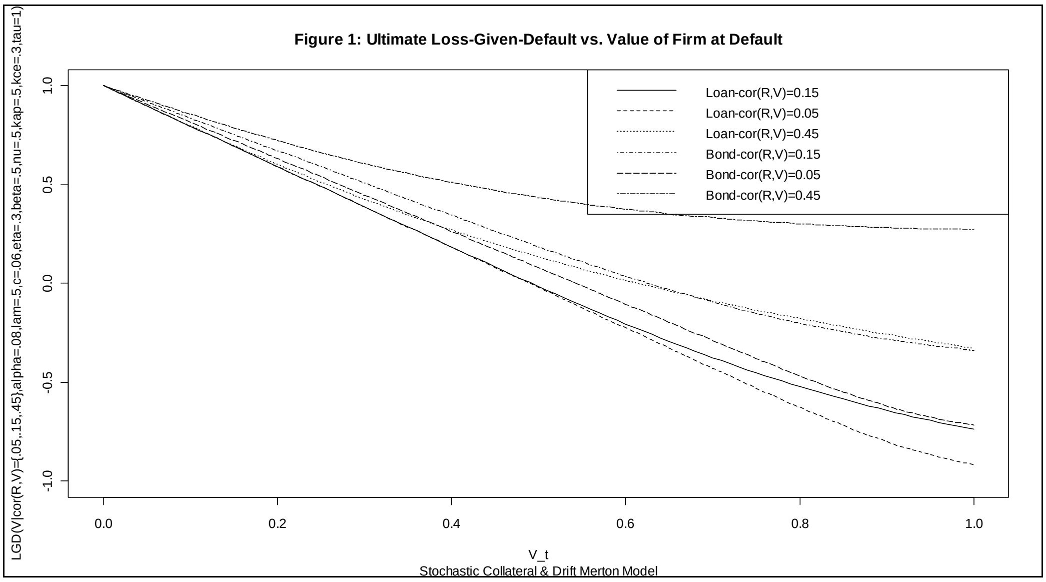
Where $\Phi_2(X, Y; \rho_{XY})$ is the bivariate normal distribution function for Brownian increments the correlation parameter is given by $\rho_{XY} = \sqrt{\frac{T_X}{T_Y}}$ for respective "expiry times" T_X and T_Y for X and Y, respectively. Note that this assumption, which is realistic in that we observe in the data that on average earlier default on the bond even if the emerges from bankruptcy or resolve a default at a single time (which in addition is random), is matter of necessity in the log-normal setting in that the bivariate normal distribution is not defined for $\rho_{XY} = \sqrt{\frac{\tau}{\tau}} = 1$ in the case that $T_X = T_Y = \tau$

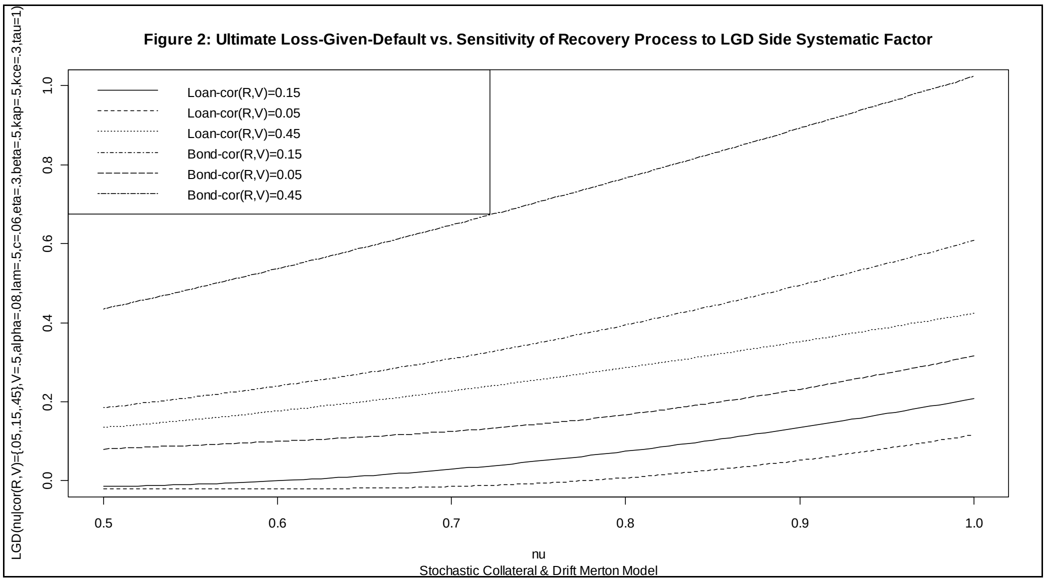
We can extend this framework to arbitrary tranches of debt, such as for a subordinated issue, in which case we follow the same procedure in order to arrive at an expression that involves trivariate cumulative normal distributions. In general, a debt issue that is subordinated to the d^{th} degree results in a pricing formula that is a linear combination of d+1 variate Gaussian distributions. These formulae become cumbersome very quickly, so for the sake of brevity we refer the interested reader to Haug (2006) for further details.

4. Comparitive statics

In this section we discuss and analyze the sensitivity of ultimate LGD in to various key parameters. In Figures 1 through 5 we examine the sensitivity of the ultimate LGD in the two-factor model of Section 3, incorporating the optimal foreclosure boundary. In Figure 1, we look at the ultimate LGD on the bond and the loan for three different settings of the factor loading of the recovery rate process on the systematic factor in the firm value processes (β = 0.05, 0.45 and 0.9), while fixing other parameters at "reasonable" values motivated by prior literature (drift in recovery α = 0.08, face value of loan λ = 0.5, coupon rate on loan c = 0.06, LGD side volatility υ = 0.3, volatility of recovery return drift process η = 0.5, speed of mean reversion in LGD return κ = 0.5, correlation between LGD side systematic factor and random factor in recovery rate drift ξ =0.3, and time-to-resolution τ =1). We observe that ultimate LGD is monotonically decreasing at increasing rate in value of the firm at default, that this increasing in the correlation between the PD and LGD side systematic factors, and that this is also uniformly higher for bonds than for loans. In Figure 2 we show the ultimate LGD as a function of the volatility in the recovery rate process attributable to the LGD side systematic factor η, fixing firm value at default at V_t = 0.5. We observe that ultimate LGD increases at an increasing rate in this parameter, that for higher correlation between firm asset value and recovery value return the LGD is higher and increases at a faster rate, and that for bonds these curves lie above and increase at a faster rate. In Figure 3 we show the ultimate LGD as a function of the volatility β in the recovery rate process attributable to the

PD side systematic factor, fixing LGD side volatility $\upsilon = 0.5$, for different firm values at default at $V_t = (0.3, 0.5, 0.8)$. We observe that ultimate LGD increases at an increasing rate in this parameter, that for lower firm asset values the LGD is higher but increases at a slower rate, and that for bonds these curves lie above and increase at a lower rate. In Figure 4 we show the ultimate LGD as a function of the volatility η of the stochastic drift in the recovery rate process, for three different settings of the factor loading of the recovery rate process on the systematic factor in the firm value processes (β = 0.05, 0.45 and 0.9). We observe that ultimate LGD in this parameter decreases at a decreasing rate, although the sensitivity is not great (especially for loans), and that as expected the curves lie above for greater PD-LGD correlation and for bonds as compared to loans. Finally, in Figure 5 we fix $\eta = 0.3$ and vary κ, the coefficient of mean reversion in the drift process for the recovery rate, and observe that ultimate LGD is increasing in this parameter, at a decreasing rate and having a discontinuity for these parameter settings; and as expected, for higher levels of default and recovery correlation, or for bonds as compared to loans, the curves lie everywhere above.





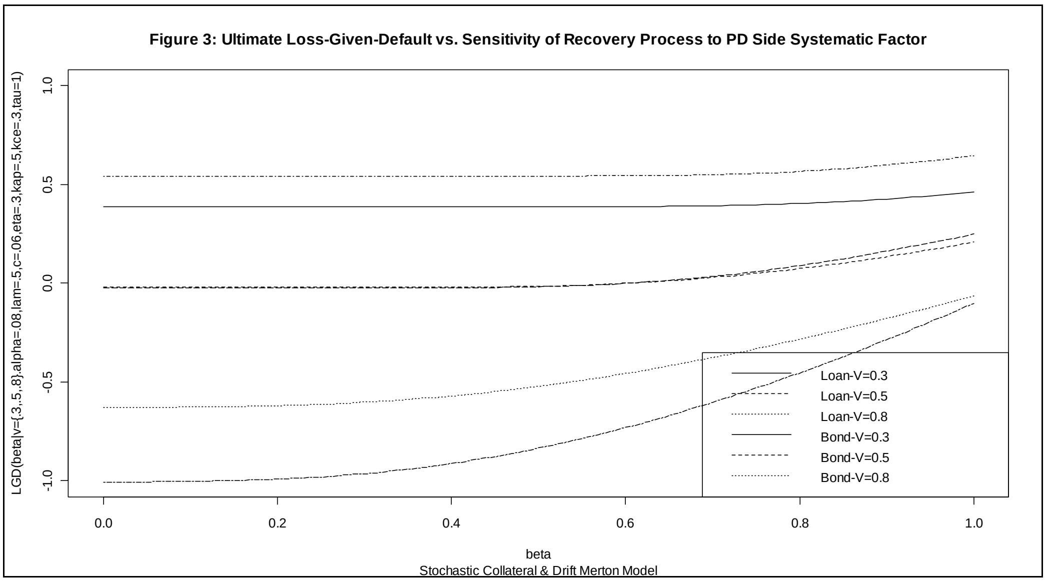


Figure 3: Ultimate Loss-Given-Default vs. Sensitivity of Recovery Process to PD Side Systematic Factor
LGD(beta|v={.3,.5,.8},alpha=.08,lam=.5,c=.06,eta=.3,kap=.5,kce=.3,tau=1)
Loan-V=0.3
Loan-V=0.5
Loan-V=0.8
Bond-V=0.3
Bond-V=0.5
Bond-V=0.8
beta
Stochastic Collateral & Drift Merton Model

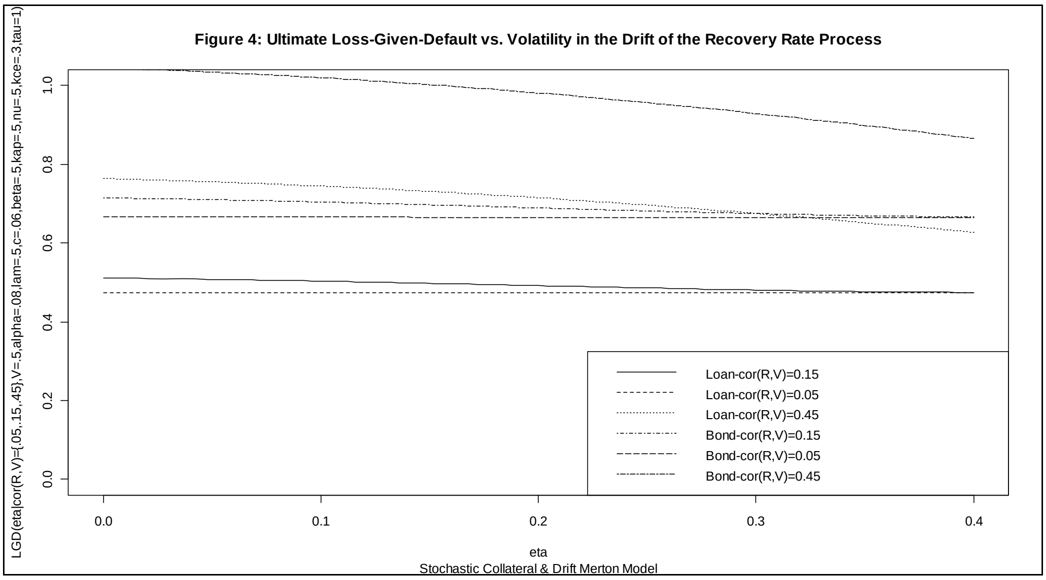


Figure 4: Ultimate Loss-Given-Default vs. Volatility in the Drift of the Recovery Rate Process
LGD(eta|cor(R,V)={.05,.15,.45},V=.5,alpha=.08,lam=.5,c=.06,beta=.5,kap=.5,nu=.5,kce=.3,tau=1)
Loan-cor(R,V)=0.15
Loan-cor(R,V)=0.05
Loan-cor(R,V)=0.45
Bond-cor(R,V)=0.15
Bond-cor(R,V)=0.05
Bond-cor(R,V)=0.45
eta
Stochastic Collateral & Drift Merton Model

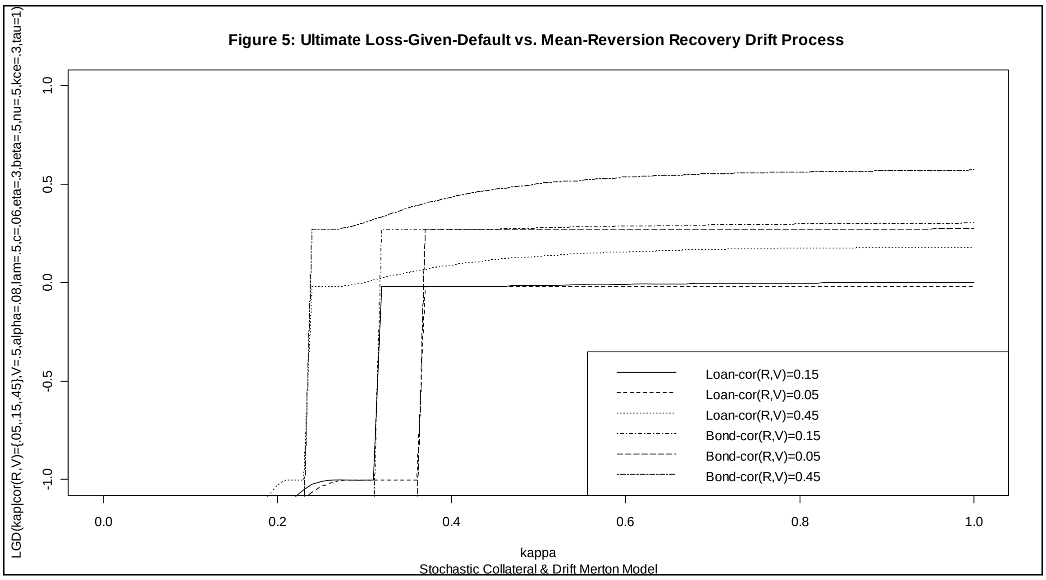


Figure 5: Ultimate Loss-Given-Default vs. Mean-Reversion Recovery Drift Process
LGD(kap|cor(R,V)={.05,.15,.45},V=.5,alpha=.08,lam=.5,c=.06,eta=.3,beta=.5,nu=.5,kce=.3,tau=1)
Loan-cor(R,V)=0.15
Loan-cor(R,V)=0.05
Loan-cor(R,V)=0.45
Bond-cor(R,V)=0.15
Bond-cor(R,V)=0.05
Bond-cor(R,V)=0.45
kappa
Stochastic Collateral & Drift Merton Model

5. Empirical analysis: calibration of models

In this section we describe our strategy for estimating parameters of the different models for LGD by *full-information maximum likelihood* (FIML.) This involves a consideration of the LGD implied in the market at time of default t_i^D for the i[th] instrument in recovery segment s, denoted LGD_{i,s,t_i^D}. This is the expected, discounted ultimate loss-given-default LGD_{i,s,t_i^E} at time of emergence t_i^E as given by any of our models m, $LGD_{s,m}^P\left(\boldsymbol{\theta}_{s,m}\right)$ over the resolution period $t_{i,s}^E - t_{i,s}^D$:

$$LGD_{i,s,t_i^D} = \frac{E_t^P\left[LGD_{i,s,t_i^E}\right]}{\left(1+r_{i,s}^D\right)^{t_{i,s}^E - t_{i,s}^D}} = LGD_{s,m}^P\left(\boldsymbol{\theta}_{s,m}\right) \tag{5.1}$$

where $\boldsymbol{\theta}_{s,m}$ is the parameter vector for segment s under model m, expectation is taken with respect to physical measure P, discounting is at risk adjusted rate appropriate to the instrument $r_{i,s}^D$ and it is assumed that the time-to-resolution $t_{i,s}^E - t_{i,s}^D$ is known.

In order to account for the fact that we cannot observe expected recovery prices ex ante, as only by coincidence would they coincide with expectations, we invoke market rationality to postulate that for a segment homogenous with respect to recovery risk the difference between expected and average realized recoveries should be small. We formulate this by defining the normalized forecast error as:

$$\tilde{\varepsilon}_{i,s} \equiv \frac{LGD_{s,m}^P\left(\boldsymbol{\theta}_{s,m}\right) - LGD_{i,s,t_i^E}}{LGD_{i,s,t_i^D} \times \sqrt{t_{i,s}^E - t_{i,s}^D}} \tag{5.2}$$

This is the forecast error as a proportion of the LGD implied by the market at default (a "unit-free" measure of recovery uncertainty) and the square root of the time-to-resolution. This is a mechanism to control for the likely increase in uncertainty with time-to-resolution, which effectively puts more weight on longer resolutions, increasing the estimate of the loss-severity. The idea behind this is that more information is revealed as the emergence point is approached, hence a decrease in risk. Alternatively, we can analyze $\varepsilon_{i,s} \equiv \frac{LGD_{s,m}^P\left(\boldsymbol{\theta}_{s,m}\right) - LGD_{i,s,t_i^E}}{LGD_{i,s,t_i^D}}$, the forecast error that is non-time adjusted, and argue that its standard error is proportional to $\sqrt{t_{i,s}^E - t_{i,s}^D}$, which is consistent with an economy in which information is revealed uniformly and independently through time (Miu and Ozdemir, 2005). Assuming that the errors $\tilde{\varepsilon}_{i,s}$ in (5.2) are standard normal,[8] we may use *full-information maximum likelihood* (FIML), by maximizing the *log-likelihood* (LL) function:

[8] If the errors are i.i.d and from symmetric distributions, then we can still obtain consistent estimates through ML, which has the interpretations as the quasi-ML estimator.

$$
\begin{aligned}
\widehat{LGD}^{P}_{s,m} &= \underset{\boldsymbol{\theta}_{s,m}}{\arg\max}\, LL = \underset{\boldsymbol{\theta}_{s,m}}{\arg\max} \sum_{i=1}^{N_s^D} \log\left[\phi\left(\tilde{\varepsilon}_{i,s}\left(\boldsymbol{\theta}_{s,m}\right)\right)\right] \\
&= \underset{\boldsymbol{\theta}_{s,m}}{\arg\max} \sum_{i=1}^{N_s^D} \log\left[\phi\left(\frac{LGD^{P}_{s,m}\left(\boldsymbol{\theta}_{s,m}\right) - LGD_{i,s,t_i^E}}{LGD_{i,s,t_i^D} \times \sqrt{t_{i,s}^E - t_{i,s}^D}}\right)\right]
\end{aligned}
\tag{5.3}
$$

This turns out to be equivalent to minimizing the squared normalized forecast errors:

$$
\widehat{LGD}^{P}_{s,m} = \underset{\boldsymbol{\theta}_{s,m}}{\arg\min}\left\{\sum_{i=1}^{N_s} \frac{1}{t_{i,s}^E - t_{i,s}^D}\left(\frac{LGD^{P}_{s,m}\left(\boldsymbol{\theta}_{s,m}\right) - LGD_{i,s,t_i^E}}{LGD_{i,s,t_i^D}}\right)^2\right\} = \underset{\boldsymbol{\theta}_{s,m}}{\arg\min}\left\{\sum_{i=1}^{N_s} \tilde{\varepsilon}_{i,s,m}{}^2\right\}
\tag{5.4}
$$

We may derive a measure of uncertainty of our estimate by the ML standard errors from the Hessian matrix evaluated at the optimum:

$$
\hat{\boldsymbol{\Sigma}}_{\hat{\boldsymbol{\theta}}_{s,m}} = \left.\left[-\frac{\partial^2 LL}{\partial \boldsymbol{\theta}_{s,m} \partial \boldsymbol{\theta}^T_{s,m}}\right]^{-\frac{1}{2}}\right|_{\boldsymbol{\theta}_{s,m} = \hat{\boldsymbol{\theta}}_{s,m}}
\tag{5.5}
$$

6. Data and estimation results

We summarize basic characteristics of our data-set in Tables 1 and 2, and the maximum likelihood estimates are shown in Table 3. These are based upon our analysis of defaulted bonds and loans in the Moody's Ultimate Recovery (MURD™) database release as of August, 2009. This contains the market values of defaulted instruments at or near the time of default,[9] as well as the values of such pre-petition instruments (or of instruments received in settlement) at the time of default resolution. This database is largely representative of the U.S. large-corporate loss experience, from the mid-1980's to the present, including most of the major corporate bankruptcies occurring in this period.

Table 1 shows summary statistics of various quantities of interest according to instrument type (bank loan, bond, term loan or revolver) and default type (bankruptcy under Chapter 11 or out-of-court renegotiation). First, we take the annualized return or yield on defaulted debt from the date of default (bankruptcy filing or distressed renegotiation date) to the date of resolution (settlement of renegotiation or emergence from Chapter 11), henceforth abbreviated as "RDD". Second, the trading price at default implied LGD ("TLGD"), or par minus the trading price of defaulted debt at the time of default (average 30-45 days after default) as a percent of par. Third, our measure of ultimate loss severity, the dollar loss-given-default on the debt instrument at emergence from bankruptcy or time of final settlement ("ULGD"), computed as par minus either values of pre-petition or settlement instruments at resolution. We also summarize two additional variables in Table 1, the total instrument outstanding at default, and the time in years from the instrument default date to the time of ultimate recovery.

9 This an average of trading prices from 30 to 45 days following the default event. A set of dealers is polled every day and the minimum /maximum quote is thrown out. This is done by experts at Moody's.

The preponderance of this sample is made up of bankruptcies as opposed to out-of-court settlements, 1,322 out of a total of 1,398 instruments. We note that out-of-court settlements have lower LGDs by either the trading or ultimate measures, 37.7% and 33.8%, as compared to Chapter 11's, 55.7% and 51.6%, respectively; and the heavy weight of bankruptcies are reflected in how close the latter are to the overall averages, 54.7% and 50.6% for TLGD and ULGD, respectively. Interestingly, not only do distressed renegotiations have lower loss severities, but such debt performs better over the default period than bankruptcies, RDD of 37.3% as compared to 28.1%, as compared to an overall RDD of 28.6%. We also note that the TLGD is higher than the ULGD by around 5% across default and instrument types, 55.7% (37.7%) as compared to 51.6% (33.8%) for bankruptcies (renegotiations). We also see that loans have better recoveries by both measures as well higher returns on defaulted debt, respective average TLGD, ULGD and RDD 52.5%, 49.3% and 32.2%. I

In Table 2 we summarize ULGD, TLGD and RDD by major collateral categories and seniority classes. We observe for this sample that either LGD measure appears to weakly exhibit the usual decreasing pattern observed in the literature with respect to higher seniority class, but this relationship is not consistent with respect to collateral categories. On the other hand, while also not monotonic, we a somewhat stronger relationship for RDD, as these tend to be higher for either better secured or more highly ranked instruments. We have average TLGD (ULGD) of 53.3% (49.3%), 51.6% (35.0%), 56.0% (38.0%), 58.5% (36.5%) and 65.8% (33.5%) for Revolving Credit/Term Loan, Senior Secured Bonds, Senior Unsecured Bonds, Senior Subordinated Bonds and Junior Subordinated Bonds, respectively. The corresponding averages of RDD in descending order of seniority class are 32.2%, 36.6%, 23.8%, 33.2% and 15.6% - an overall decreasing albeit non-monotonic pattern. On the other hand, for this particular sample and segmentation of collateral codes, we fail to see much of a rank ordering as we might have expected. We have average TLGD (ULGD) of 66.5% (65.0%), 41.6% (32.9%), 50.6% (47.6%), 61.6% (48.6%), 59.3% (59.4%) and 57.4% (51.46%) for Cash, Accounts Receivables & Guarantees, Inventory/Most Assets & Equipment, All Assets & Real Estate, Non-Current Assets & Capital Stock, PPE/Second Lien and Unsecured/Other Illiquid Collateral, respectively. Even just focusing upon the split between secured and unsecured, we fail to see much (any) of a difference in TLGD (ULGD), 57.58% vs. 53.40% (37.69% vs. 36.13%), respectively. The corresponding averages of RDD in descending order of collateral quality are: 22.6%, 33.2%, 33.8%, 46.2%, 29.0% and 24.1% - a humped shaped pattern. However, RDD is higher for secured as compared to unsecured, 34.5% vs. 3.6%, respectively.

Table 1

Characteristics of loss-given-default and return on defaulted debt observations by default and instrument type

(Moody's Ultimate Recovery Database 1987-2009)

		Bankruptcy			Out-of-Court			Total		
		Count	**Average**	**Standard Error of the Mean**	**Count**	**Average**	**Standard Error of the Mean**	**Count**	**Average**	**Standard Error of the Mean**
Bonds and Term Loans	Return on Defaulted Debt[1]	1072	28.32%	3.47%	59	45.11%	19.57%	1131	29.19%	3.44%
	LGD at Default[2]		55.97%	0.96%		38.98%	3.29%		55.08%	0.93%
	Discounted LGD[3]		51.43%	1.15%		33.89%	3.05%		50.52%	1.10%
	Time-to-Resolution[4]		1.7263	0.0433		0.0665	0.0333		1.6398	0.0425
	Principal at Default[5]		207'581	9'043		416'751	65'675		218'493	9'323
Bonds	Return on Defaulted Debt[1]	837	25.44%	3.75%	47	44.22%	21.90%	884	26.44%	3.74%
	LGD at Default[2]		57.03%	1.97%		37.02%	5.40%		55.96%	1.88%
	Discounted LGD[3]		52.44%	1.30%		30.96%	3.00%		51.30%	1.25%
	Time-to-Resolution[4]		1.8274	0.0486		0.0828	0.0415		1.7346	0.0424
	Principal at Default[5]		214'893	11'148		432'061	72'727		226'439	11'347
Revolvers	Return on Defaulted Debt[1]	250	26.93%	7.74%	17	10.32%	4.61%	267	25.88%	7.26%
	LGD at Default[2]		54.37%	1.96%		33.35%	8.10%		53.03%	1.93%
	Discounted LGD[3]		52.03%	2.31%		33.33%	7.63%		50.84%	2.23%
	Time-to-Resolution[4]		1.4089	0.0798		0.0027	0.0000		1.3194	0.0776
	Principal at Default[5]		205'028	19'378		246'163	78'208		207'647	18'786
Loans	Return on Defaulted Debt[1]	485	32.57%	5.71%	29	26.161 %	18.872 %	514	32.21%	5.49%
	LGD at Default[2]		53.31%	9.90%		38.86%	7.22%		52.50%	3.21%
	Discounted LGD[3]		50.00%	1.68%		38.31%	5.79%		49.34%	2.25%
	Time-to-Resolution[4]		1.3884	0.0605		0.0027	0.0000		1.3102	0.0816
	Principal at Default[5]		193'647	11'336		291'939	78'628		199'192	16'088

Table 1 (cont)

Characteristics of loss-given-default and return on defaulted debt observations by default and instrument type

(Moody's Ultimate Recovery Database 1987-2009)

		Bankruptcy			Out-of-Court			Total		
		Count	Average	Standard Error of the Mean	Count	Average	Standard Error of the Mean	Count	Average	Standard Error of the Mean
Total	Return on Defaulted Debt[1]	1322	28.05%	3.17%	76	37.33%	15.29%	1398	28.56%	3.11%
	LGD at Default[2]		55.66%	0.86%		37.72%	3.12%		54.69%	0.84%
	Discounted LGD[3]		51.55%	1.03%		33.76%	2.89%		50.58%	0.99%
	Time-to-Resolution[4]		1.6663	0.0384		0.0522	0.0260		1.5786	0.0376
	Principal at Default[5]		207'099	8'194		378'593	54'302		216'422	8'351

[1] Annualized return or yield on defaulted debt from the date of default (bankruptcy filing or distressed renegotiation date) to the date of resolution (settlement of renegotiation or emergence from Chapter 11). [2] Par minus the price of defaulted debt at the time of default (average 30-45 days after default) as a percent of par. [3] The ultimate dollar loss-given-default on the defaulted debt instrument = 1 - (total recovery at emergence from bankruptcy or time of final settlement)/(outstanding at default). Alternatively, this can be expressed as (outstanding at default - total ultimate loss)/(outstanding at default). [4] The total instrument outstanding at default. [5] The time in years from the instrument default date to the time of ultimate recovery.

Table 2

Loss-given-default by seniority ranks and collateral types

(Moody's Ultimate Recovery Database 1987-2009)

Collateral Type			Cash, Accounts Receivables & Guarantees	Inventory, Most Assets & Equipment	All Assets & Real Estate	Non-Current Assets & Capital Stock	PPE & Second Lien	Unsecured & Other Illiquid Collateral	Total Unsecured	Total Secured	Total Collateral
Revolving Credit / Term Loan	Count		39	8	367	38	29	33	32	482	514
	LGD at Default[1]	Average	66.81%	46.60%	51.95%	59.94%	55.02%	45.63%	46.25%	53.79%	53.31%
		Standard Error	4.44%	11.79%	1.70%	5.27%	6.08%	5.07%	5.20%	1.47%	1.42%
	Ultimate LGD[2]	Average	64.38%	56.03%	48.58%	50.62%	56.53%	30.70%	31.78%	50.51%	49.34%
		Standard Error	5.09%	13.85%	1.91%	6.10%	6.88%	6.17%	5.20%	1.47%	1.42%
	Return on Defaulted Debt[3]	Average	22.57%	-5.80%	33.49%	35.68%	46.07%	22.39%	19.77%	33.03%	32.21%
		Standard Error	18.20%	30.27%	6.89%	15.01%	27.64%	8.12%	7.93%	5.83%	5.49%
Senior Secured Bonds	Count		2	38	41	35	7	142	3	139	142
	LGD at Default[1]	Average	61.50%	40.19%	36.02%	62.99%	61.24%	51.67%	50.73%	51.59%	51.57%
		Standard Error	36.50%	5.50%	5.03%	4.71%	11.63%	2.48%	23.79%	2.76%	2.74%
	Ultimate LGD[2]	Average	76.81%	23.87%	36.67%	46.70%	60.32%	49.68%	50.15%	34.88%	35.04%
		Standard Error	19.39%	3.90%	5.61%	5.71%	12.68%	3.19%	28.95%	2.96%	2.94%
	Return on Defaulted Debt[3]	Average	23.86%	47.53%	35.03%	55.99%	14.33%	17.44%	-27.66%	38.02%	36.63%
		Standard Error	40.63%	7.18%	22.04%	20.10%	27.41%	6.34%	36.65%	9.05%	8.92%
Senior Unsecured Bonds	Count		0	0	1	0	1	459	452	9	461
	LGD at Default[1]	Average	0.00%	0.00%	85.00%	N/A	80.00%	55.83%	55.94%	56.63%	55.96%
		Standard Error	N/A	N/A	N/A	N/A	N/A	1.42%	1.43%	10.36%	1.42%
	Ultimate LGD[2]	Average	0.00%	0.00%	78.76%	N/A	74.25%	48.33%	38.14%	32.03%	38.00%
		Standard Error	N/A	N/A	N/A	N/A	N/A	1.78%	1.79%	10.68%	1.77%
	Return on Defaulted Debt[3]	Average	0.00%	0.00%	86.47%	n	119.64%	23.40%	23.71%	25.62%	23.75%
		Standard Error	N/A	N/A	N/A	N/A	N/A	4.80%	4.86%	22.61%	4.78%
Senior Subordinated Bonds	Count		0	0	1	0	1	159	158	3	161
	LGD at Default[1]	Average	0.00%	N/A	85.00%	N/A	90.50%	58.09%	57.98%	83.46%	58.48%
		Standard Error	N/A	N/A	N/A	N/A	N/A	2.48%	2.50%	4.58%	2.47%
	Ultimate LGD[2]	Average	N/A	N/A	74.72%	N/A	97.74%	54.51%	36.50%	40.47%	36.46%
		Standard Error	N/A	N/A	N/A	N/A	N/A	2.89%	2.90%	23.36%	2.87%
	Return on Defaulted Debt[3]	Average	0.00%	N/A	57.45%	N/A	-45.98%	33.57%	31.01%	150.30%	33.23%
		Standard Error	N/A	N/A	N/A	N/A	N/A	10.44%	10.18%	147.62%	10.32%

Table 2 (cont)

Loss-given-default by seniority ranks and collateral types

(Moody's Ultimate Recovery Database 1987-2009)

Collateral Type			Cash, Accounts Receivables & Guarantees	Inventory, Most Assets & Equipment	All Assets & Real Estate	Non-Current Assets & Capital Stock	PPE & Second Lien	Unsecured & Other Illiquid Collateral	Total Unsecured	Total Secured	Total Collateral
Junior Subordinated Bonds	Count		0	1	0	0	0	119	117	3	120
	LGD at Default[1]	Average	N/A	27.33%	0.00%	N/A	N/A	66.15%	66.58%	37.42%	65.81%
		Standard Error	N/A	N/A	N/A	N/A	N/A	2.50%	2.48%	22.25%	2.50%
	Ultimate LGD[2]	Average	N/A	20.15%	0.00%	N/A	N/A	65.36%	33.62%	32.77%	33.54%
		Standard Error	N/A	N/A	N/A	N/A	N/A	3.06%	3.11%	18.92%	3.06%
	Return on Defaulted Debt[3]	Average	N/A	72.13%	0.00%	N/A	N/A	15.11%	15.74%	9.49%	15.59%
		Standard Error	N/A	N/A	N/A	N/A	N/A	10.93%	11.11%	31.36%	10.85%
Total Instruments	Count		41	28	407	79	66	777	762	636	1398
	LGD at Default[1]	Average	66.53%	41.57%	50.55%	61.56%	59.31%	57.41%	57.58%	53.40%	55.66%
		Standard Error	4.41%	6.18%	1.63%	3.39%	3.86%	1.09%	1.10%	1.28%	0.84%
	Ultimate LGD[2]	Average	64.98%	32.93%	47.60%	48.58%	59.43%	51.46%	37.69%	36.13%	36.99%
		Standard Error	4.90%	5.99%	1.82%	3.99%	4.28%	1.35%	1.37%	1.43%	0.99%
	Return on Defaulted Debt[3]	Average	22.63%	33.17%	33.82%	46.22%	28.96%	24.12%	34.46%	23.63%	28.56%
		Standard Error	17.36%	11.74%	6.56%	12.02%	13.90%	3.94%	3.97%	4.89%	3.11%

[1] Par minus the price of defaulted debt at the time of default (average 30-45 days after default) as a percent of par. [2] The ultimate dollar loss-given-default on the defaulted debt instrument = 1 - (total recovery at emergence from bankruptcy or time of final settlement)/(outstanding at default). Alternatively, this can be expressed as (outstanding at default - total ultimate loss)/(outstanding at default). [3] Annualized return or yield on defaulted debt from the date of default (bankruptcy filing or distressed renegotiation date) to the date of resolution (settlement of renegotiation or emergence from Chapter 11).

In Table 3 we present the full-information maximum likelihood estimation (FIML) results of the leading model for ultimate LGD derived in this paper, the two-factor structural model of ultimate loss-given-default, with systematic recovery risk and random drift (2FSM-SR&RD) on the recovery process.[10] The model is estimated along with the optimal foreclosure boundary constraint.

We first discuss the MLE point estimates of the parameters governing the firm value process and default risk, or the "PD-side". Regarding the parameter σ, which is the volatility of the firm-value process governing default, we observe that estimates are decreasing in seniority class, ranging from 9.1% to 4.3% from subordinated bonds to senior loans. As standard errors range in 1% to 2%, increasing in seniority rank, these differences across seniority classes and models are generally statistically significant. Regarding the MLE point estimates of the parameter μ, which is the drift of the firm-value process governing default, we observe estimates are increasing in seniority class, ranging from 9.6% to 18.6% from subordinated bonds to loans, respectively.

[10] Estimates for the baseline Merton structural model (BMSM) and for the Merton structural model with stochastic drift (MSM-SD) are available upon request.

Table 3

Full information maximum likelihood estimation of option theoretic two-factor structural model of ultimate loss-given-default with optimal foreclosure boundary, systematic recovery risk and random drift in the recovery process

(Moody's Ultimate Recovery Database 1987-2009)

Recovery Segment		Parameter	σ[1]	μ[2]	β[3]	v[4]	σR[5]	πRβ[6]	πRv[7]	(βσ)0.5	κα[8]	α[9]	ηα[10]	ς[11]
Seniority Class	Revolving Credit / Term Loan	Est.	4.32%	18.63%	18.16%	36.83%	41.06%	19.55%	80.45%	12.82%	3.96%	37.08%	48.85%	20.88%
		Std. Err.	0.5474%	0.9177%	0.7310%	1.3719%				0.4190%	0.0755%	4.2546%	3.2125%	0.9215%
	Senior Secured Bonds	Est.	5.47%	16.99%	16.54%	30.41%	34.62%	22.83%	77.17%	11.64%	4.40%	33.66%	44.43%	18.99%
		Std. Err.	0.5314%	0.8613%	0.6008%	1.3104%				0.7448%	0.0602%	3.5085%	2.6903%	0.8297%
	Senior Unsecured Bonds	Est.	6.82%	14.16%	13.82%	24.38%	28.02%	24.30%	75.70%	9.71%	5.50%	28.07%	37.04%	15.83%
		Std. Err.	0.5993%	1.0813%	1.3913%	1.9947%				0.6165%	0.0281%	2.8868%	2.2441%	0.6504%
	Senior Subordinated Bonds	Est.	8.19%	11.33%	12.02%	17.35%	21.11%	32.43%	67.57%	7.76%	4.42%	22.45%	29.68%	12.69%
		Std. Err.	0.6216%	1.0087%	1.0482%	1.0389%				0.9775%	0.0181%	2.0056%	2.0132%	1.0016%
	Subordinated Bonds	Est.	9.05%	9.60%	10.24%	12.37%	16.06%	40.66%	59.34%	5.97%	3.34%	18.80%	18.69%	9.43%
		Std. Err.	0.6192%	1.0721%	1.0128%	1.0771%				0.9142%	0.0106%	2.0488%	2.0014%	1.0142%
Value Log-Likelihood Function			-371.09											
Degrees of Freedon			1391											
P-Value of Likelihood Ratio Statistic			4.69E-03											
In-Sample / Time Diagnostic Statistics	Area Under ROC Curve		93.14%											
	Komogorov-Smirnov Stat. (P-Val.)		2.14E-08											
	McFadden Pseudo R-Squared		72.11%											
	Hoshmer-Lemeshow Chi-Squared (P-Values)		0.63											

[1] The volatility of the firm-value process governing default. [2] The drift of the firm-value process governing default. [3] The sensitivity of the recovery-rate process to the systematic governing default in (or the component of volatility in the recovery process due to PD-side systematic risk). [4] The sensitivity of the recovery-rate process to the systematic governing collateral value (or the component of volatility in the recovery process due to LGD-side systematic risk). [5] The total volatility of the recovery rate process: sqrt(β^2+v^2). [6] Component of total recovery variance attributable to PD-side (asset value) uncertainty: $\beta^2/(\beta^2+v^2)$. [7] Component of total recovery variance attributable to LGD-side (collateral value) uncertainty: $v^2/(\beta^2+v^2)$. [8] The speed of the mean-reversion in the random drift in the recovery rate process. [9] The long-run mean of the random drift in the recovery arte process. [10] The volatility of the random drift in the recovery rate process. [11] The correlation of the random processes in drift of and the level of the recovery rate process.

These too are statistically significant across seniorities. The fact that we are observing different estimates of a single firm value process across seniorities is evidence that models which attribute identical default risk across different instrument types are misspecified – in fact, we are measuring lower default risk (i.e., lower asset value volatility and greater drift in firm-value) in loans and senior secured bonds as compared to unsecured and subordinated bonds.

A key result regards the magnitudes and composition of the components of recovery volatility across maturities inferred from the model calibration. The MLE point estimates of the parameter β, the sensitivity of the recovery-rate process to the systematic factor governing default (or due to PD-side systematic risk), increases in seniority class, from 10.2% for subordinated bonds to 18.2% senior bank loans. On the other hand, estimates of the parameter ν, the sensitivity of the recovery-rate process to the systematic factor governing collateral value (or due to LGD-side systematic risk), are greater than β across seniorities, and similarly increases in from 12.4% for subordinated bonds to 36.8% for bank loans. This monotonic increase in both β and ν as we move up in the hierarchy of the capital structure from lower to higher ranked instruments has the interpretation of a greater sensitivity in the recovery rate process attributable to both systematic risks, implying that total recovery volatility $\sigma_R = \sqrt{\beta^2 + \upsilon^2}$ increases from higher to lower ELGD instruments, from 16.1% for subordinated bonds to 41.1% for senior loans. However, we see that the *proportion* of the total recovery volatility attributable to systematic risk in collateral (firm) value, or the LGD (PD) side, is increasing (decreasing) in seniority from 59.3% to 80.5% (40.7% to 19.6%) from subordinated bonds to senior bank loans. Therefore, more senior instruments not only exhibit greater recovery volatility than less senior instruments, but a larger component of this volatility is driven by the collateral rather than the asset value process.

The next set of results concern the random drift in the recovery rate process. The MLE point estimates of the parameter κ_α, the speed of the mean-reversion in, is hump-shaped in seniority class, ranging from 3.3% subordinated bonds, to 5.5% for senior unsecured bonds, to 4.0% for loans, respectively. Estimates of the parameter α, the long-run mean of the random drift in the recovery rate process, increase in seniority class from 18.8% for subordinated bonds to 37.1% for senior bank loans. This monotonic increase in α as we move from lower to higher ranked instruments has the interpretation of greater expected return of the recovery rate process inferred from lower ELGD (or greater expected recovery) instruments as we move up in the hierarchy of the capital structure. We see that the volatility of the random drift in the recovery rate process η_α, increases in seniority class, ranging from 18.7% to 48.9% from subordinated bonds to senior loans, respectively. The monotonic increase in η_α as we move from lower to higher ranked instruments has the interpretation of greater volatility in expected return of the recovery rate process inferred from lower ELGD (or greater expected recovery) instruments as we move up in the hierarchy of the capital structure. Finally, estimates of the parameter ζ, the correlation of the random processes in drift of and the level of the recovery rate process, increases in seniority class from 9.4% for subordinated bonds to 20.9% for senior bank loans.

Finally with respect to parameter estimates, regarding the MLE point estimates of the correlation between the default and recovery rate processes $\sqrt{\beta\sigma}$ in the 2FSM-SR&RD, we observe estimates are increasing in seniority class, ranging from 6.0% to 12.8% from subordinated bonds to loans, respectively.

We conclude this section by discussing the quality of the estimates and model performance measures. Across seniority classes, parameter estimates are all statistically significant, and the magnitudes of such estimates are in general distinguishable across segments at conventional significance levels. The likelihood ratio statistic indicates that we can reject the null hypothesis that all parameter estimates are equal to zero across all ELGD segments, a p-value of 4.7e-3. We also show various diagnostics that assess in-sample fit, which show that the model performs well-in sample. The *area under receiver operating characteristic curve* (AUROC) of 93.1% is high by commonly accepted standards, indicating a good ability of the model to discriminate between high and low LGD defaulted instruments. Another test of discriminatory ability of the models is the *Kolmogorov-Smirnov* (KS) statistic, the very small p-value 2.1e-8 indicating adequate separation in the distributions of the low and high LGD instruments in the model.[11] We also show two tests of predictive accuracy, which is the ability of the model to accurately quantify a level of LGD. The *McFadden psuedo r-squared* (MPR2) is high by commonly accepted standards, 72.1%, indicating a high rank-order correlation between model and realized LGDs of defaulted instruments. Another test of predictive accuracy of the models is the *Hoshmer-Lemeshow* (HL) statistic, high p-values of 0.63 indicating high accuracy of the model to forecast cardinal LGD.

7. Downturn LGD

In this section we explore the implications of our model with respect to downturn LGD in the 2FSM-SR&RD. This is a critical component of the quantification process in the Basel II advanced IRB framework for regulatory capital. The Final Rule (FR) in the U.S. (OCC et al, 2007) requires banks that either wish, or are required, to qualify for treatment under the advanced approach to estimate a *downturn LGD*. We paraphrase the FR, this is an LGD estimated during an historical reference period during which default rates are elevated within an institution's loan portfolio.

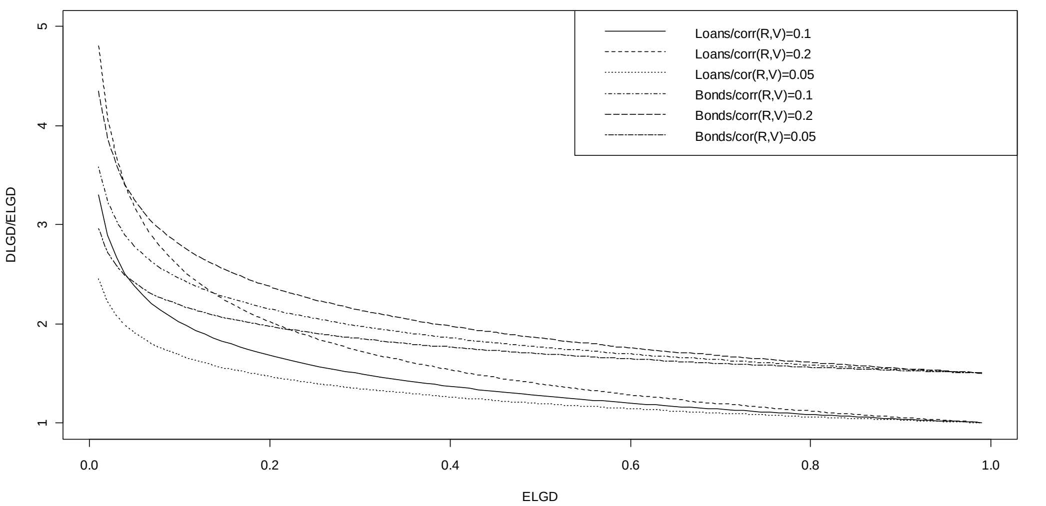
In Figures 6 through 8 we plot the ratios of the downturn LGD to the expected LGD. This is derived by conditioning on the 99.9th quantile of the PD side systematic factor in the 2FSM-SR&RD. We show this for loans and bonds, as well as for different settings of key parameters ($\sqrt{\beta\sigma}$, ν or η_α) in each plot, with other parameters set to the MLE estimates. We observe that the LGD mark-up for downturn is monotonically declining in ELGD, which is indicative of lower tail risk in recovery for lower ELGD instruments. It is also greater than unity in all cases, and approaches 1 as ELGD approaches 1. This multiple is higher for bonds than for loans, as well as for either higher PD-LGD correlation $\sqrt{\beta\sigma}$, collateral specific volatility ν or volatility in the drift of the recovery rate drift process η_α; although these differences narrow for higher ELGD. For example, in Figure 6, we see that for loans having ELGD of 15% and $\sqrt{\beta\sigma}$ = 10% (=20%), the ratio of downturn to ELGD is about 2 (2.5); but for ELGD of 50%, this is about 1.5 (1.6); and for ELGD of 80%, this about 1.2 (1.3). And for bonds having ELGD of 15% and $\sqrt{\beta\sigma}$ = 10% (=20%), the ratio of downturn to ELGD is about 2.5 (23); but for ELGD of 50%, this is about 2 (2.2) ; and for ELGD of 80%, this is about 1.6 (1.7).

[11] In these tests we take the median LGD to be the cut-off that distinguishes between a high and low realized LGD.

8. Model validation

In this final section we validate our preferred model, the 2FSM-SR&RD. In particular, we implement an out-of-sample and out-of-time analysis, on a rolling annual cohort basis for the final nine years of our sample. Furthermore, we augment this by resampling on both the training and prediction samples, a non-parametric bootstrap (Efron [1979], Efron and Tibshirani [1986], Davison and Hinkley [1997]). The procedure is as follows: the first training (or estimation) sample is established as the cohorts defaulting in the 10 years 1987-1996, and the first prediction (or validation) sample is established as the 1997 cohort. Then we resample 100,000 times with replacement from the training sample the 1987-1996 cohorts and for the prediction sample 1997 cohort, and then based upon the fitted model in the former we evaluate the model based upon the latter. Then we augment the training sample with the 1997 cohort, and establish the 1998 cohort as the prediction sample, and repeat this. This is continued until we have left the 2008 cohort as the holdout. Finally, to form our final holdout sample, we pool all of our out-of-sample resampled prediction cohorts, the 12 years running from 1997 to 2008. We then analyze the distributional properties (such as median, dispersion and shape) of the two key diagnostic statistics: the Spearman rank-order correlation for discriminatory (or classification) accuracy, and the Hoshmer-Lemeshow Chi-Squared (HLCQ) P-values for predictive accuracy, or calibration.

Figure 6: Ratio of Ultimate Downturn to Expected LGD vs. ELGD at 99.9th Percentile of PD-Side Systematic Factor Z



Stochastic Collateral & Drift Merton Model (Parameters Set to MLE Estimates for Loans & Bonds)

Figure 7: Ratio of Ultimate Downturn to Expected LGD vs. ELGD at 99.9th Percentile of PD-Side Systematic Factor Z

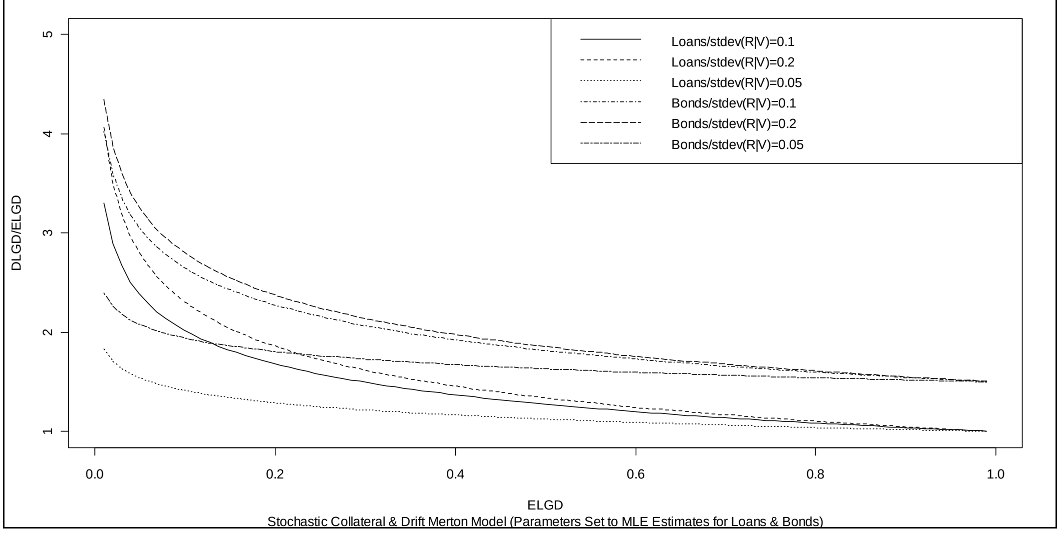
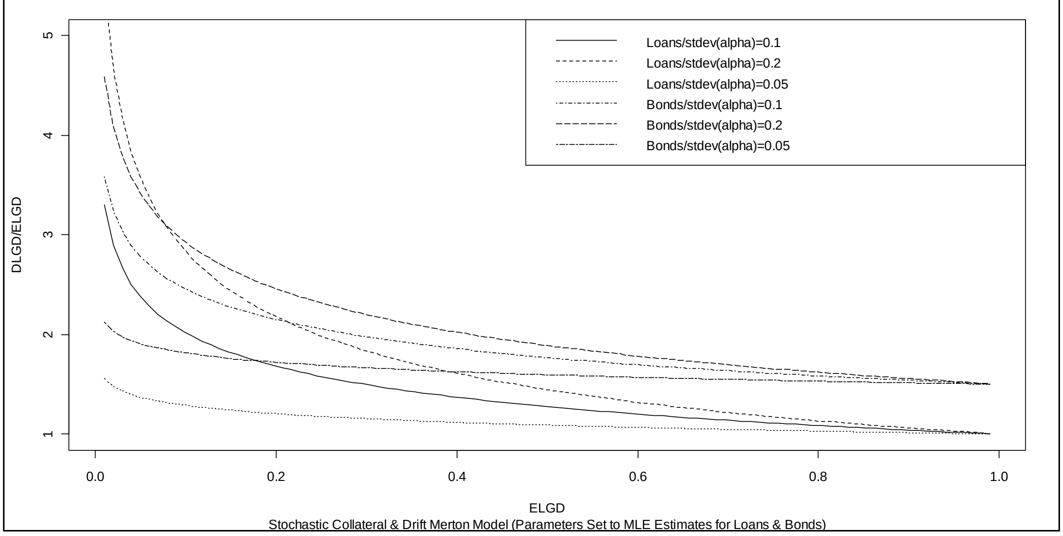


Figure 8: Ratio of Ultimate Downturn to Expected LGD vs. ELGD at 99.9th Percentile of PD-Side Systematic Factor Z



Before discussing the results, we briefly describe the two alternative frameworks for predicting ultimate LGD that are to be compared to the 2FSM-SR&RD developed in this paper. First, we implement a *full-information maximum likelihood simultaneous equation regression model* (FIMLE-SEM) for ultimate LGD, which is an econometric model built upon observations in URD at both the instrument and obligor level. FIMLE is used to model the endogeneity of the relationship between LGD at the firm and instrument levels in an internally consistent manner. This technique enables us to build a model that can help us understand some of the structural determinants of LGD, and potentially improve our forecasts of LGD. This model contains 199 observations from the URD™ with variables: long-term debt to market value of equity, book value of assets quantile, intangibles to book value of assets, interest coverage ratio, free cash flow to book value of assets, net income to net sales, number of major creditor classes, percent secured debt, Altman Z-Score, debt vintage (time since issued), Moody's 12-month trailing speculative grade default rate, industry dummy, filing district dummy and a pre-packaged bankruptcy dummy. Detailed discussion of the results can be found in Jacobs and Karagozoglu (2011). The second alternative model we consider addresses the problem of non-parametrically estimating a regression relationship, in which there are several independent variables and in which the dependent variable is bounded, as an application to the distribution of LGD. Standard non-parametric estimators of unknown probability distribution functions, whether or not conditional or not, utilize the Gaussian kernel (Silverman (1982), Hardle and Linton (1994) and Pagan and Ullah (1999)). It is well known that there exists a boundary bias with a Gaussian kernel, which assigns non-zero density outside the support on the dependent variable, when smoothing near the boundary. Chen (1999) has proposed a *beta kernel density estimator* (BKDE) defined on the unit interval [0,1], having the appealing properties of flexible functional form, a bounded support, simplicity of estimation, non-negativity and an optimal rate of convergence $n^{-4/5}$ in finite samples. Furthermore, even if the true density is unbounded at the boundaries, the BKDE remains consistent (Bouezmarni and Rolin, 2001), which is important in the context of LGD, as there are point masses (observation clustered at 0% and 100%) in empirical applications. We extend the BKDE (Renault and Scalliet, 2004) to a *generalized beta kernel conditional density estimator* (GBKDE), in which the density is a function of several independent variables, which affect the smoothing through the dependency of the beta distribution parameters upon these variables. Detailed derivation of this model can be found in Jacobs and Karagozoglu (2007), who also present a "horse-race" as herein between GBKDE the FIMLE-SEM.

Results of the model validation are shown in Table 4 and Figures 9-10. We see that while all models perform decently out-of-sample in terms of rank ordering capability, FIMLE-SEM performs the best (median = 83.2%), the GBKDE the worst (median = 72.0%), and our 2FSM-SR&RD in the middle (median = 79.1%). It is also evident from the table and figures that the better performing models are also less dispersed and exhibit less multi-modality. However, the structural model is closer in performance to the regression model by the distribution of the Pearson correlation, and indeed there is a lot of overlap in these. Unfortunately, the out-of-sample predictive accuracy is not as encouraging for any of the models, as in a sizable proportion of the runs we can reject adequacy of fit (ie p-values indicating rejection of the null of that the model fits the data it at conventional levels). The rank ordering of model performance is the same as for the Pearson statistics: FIMLE-SEM performs the best (median = 24.8%), the GBKDE the worst (median = 13.2%), and our 2FSM-SR&RD in the middle (median = 23.9%); and the structural model developed herein is comparable in out-of-sample predictive accuracy to the high-dimensional regression model. We conclude that while all models are challenged in predicting cardinal levels of ultimate LGD out-of-sample, it is remarkable that a relatively parsimonious structural model of ultimate LGD can perform so closely to a highly parameterized econometric model.

Table 4

Bootstrapped[1] out-of-sample and out-of-time classification and predictive accuracy model comparison analysis of alternative models for ultimate loss-given-default

(Moody's Ultimate Recovery Database 1987-2009)

	Test Statistic	Model	GBKDE[4]	2FSM-SR&RD[5]	FIMLE-SEM[6]
Out-of-Sample / Time 1 Year Ahead Prediction	Spearman Rank-Order Correlation[2]	Median	0.7198	0.7910	0.8316
		Standard Deviation	0.1995	0.1170	0.1054
		5th Percentile	0.4206	0.5136	0.5803
		95th Percentile	0.9095	0.9563	0.9987
	Hoshmer-Lemeshow Chi-Squared (P-Values)[3]	Median	0.1318	0.2385	0.2482
		Standard Deviation	0.0720	0.0428	0.0338
		5th Percentile	0.0159	0.0386	0.0408
		95th Percentile	0.2941	0.5547	0.5784

[1] In each run, observations are sampled randomly with replacement from the training and prediction samples, the model is estimated in the training sample and observations are classified in the prediction period, and this is repeated 100,000 times. [2] The correlation between the ranks of the predicted and realizations, a measure of the discriminatory accuracy of the model. [3] A normalized average deviation between empirical frequencies and average modelled probabilities across deciles of risk, ranked according to modelled probabilities, a measure of model fit or predictive accuracy of the model. [4] Generalized beta kernel conditional density estimator model. [5] Two-factor structural Merton systematic recovery and random drift model. [6] Full-information maximum likelihood simultaneous equation regression model. 199 observations with variables: long-term debt to market value of equity, book value of assets quantile, intangibles to book value of assets, interest coverage ratio, free cash flow to book value of assets, net income to net sales, number of major creditor classes, percent secured debt, Altman Z-Score, debt vintage (time since issued), Moody's 12-month trailing speculative grade default rate, industry dummy, filing district dummy and prepackaged bankruptcy dummy.

Fig. 9 - Densities of Pearson Correlations for LGD Prediction

100,000 Repetitions Out-of-Sample and Out-of-Time 1997-2008

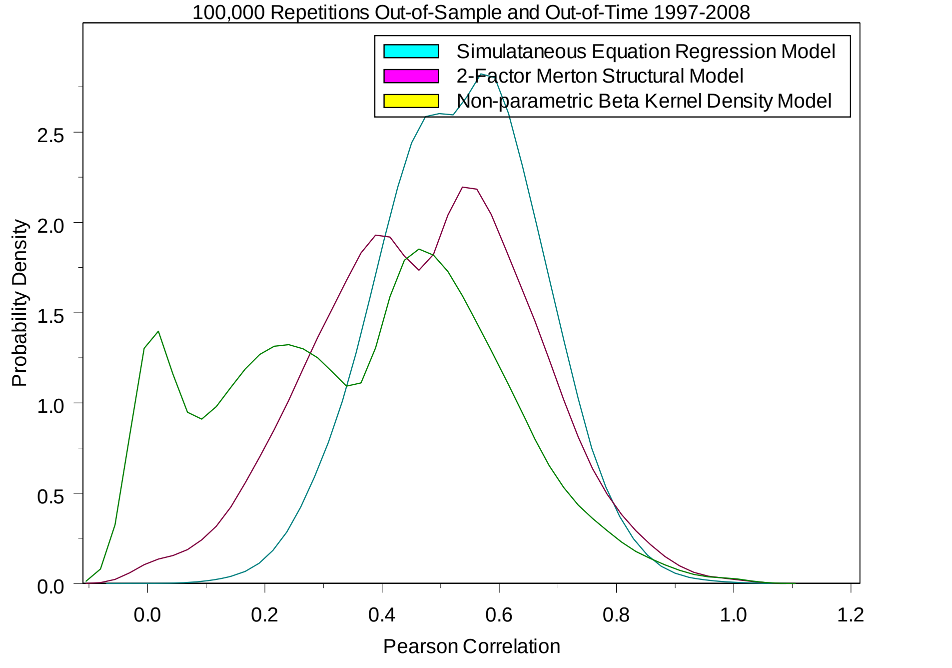
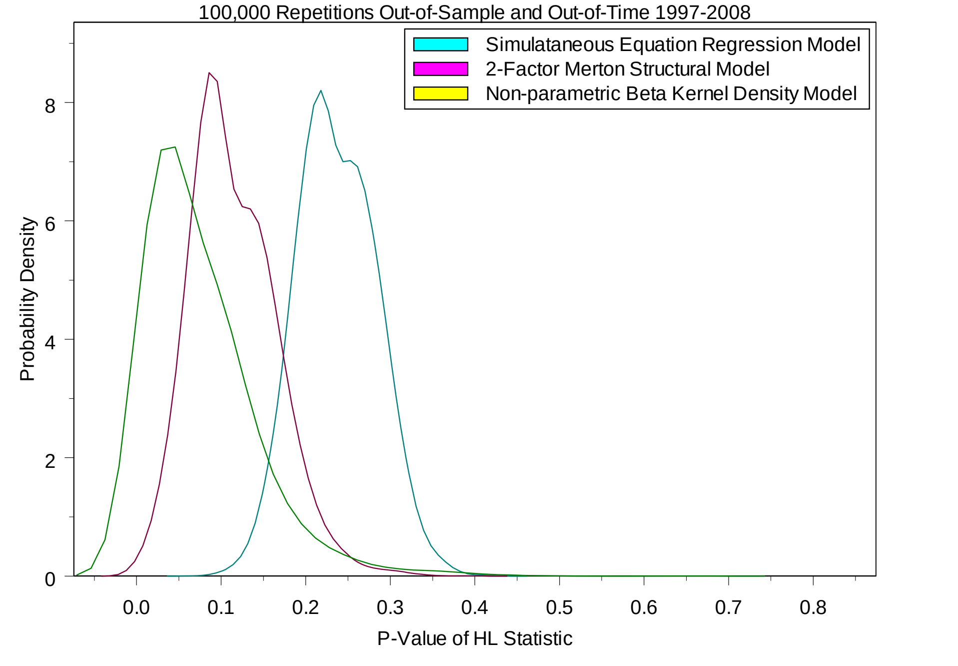


Fig.10 - Densities of Hoshmer-Lemeshow P-Values for LGD Prediction

100,000 Repetitions Out-of-Sample and Out-of-Time 1997-2008



9. Conclusions and directions for future research

In this study, we have developed a theoretical model for ultimate loss-given-default, having many intuitive and realistic features, in the structural credit risk modeling framework. Our extension admits differential seniority within the capital structure, an independent process representing a source of undiversifiable recovery risk with a stochastic drift, and an optimal foreclosure threshold. We have analyzed the comparative statics of this model and compared these to a baseline structural model. In the empirical analysis we calibrated alternative models for ultimate LGD on bonds and loans, having both trading prices at default and at resolution of default, utilizing an extensive sample of agency-rated defaulted firms in the Moody's URD™. These 800 defaults are largely representative of the US large corporate loss experience, for which we have the complete capital structures, and can track the recoveries on all instruments to the time of default to the time of resolution.

We demonstrated that parameter estimates vary significantly across models and recovery segments, finding that the estimated volatilities of the recovery rate processes and their random drifts are increasing in seniority; in particular, for first-lien bank loans as compared to senior secured or unsecured bonds. We argued that this as reflects the inherently greater risk in the ultimate recovery for higher ranked instruments having lower expected loss severities. In an exercise highly relevant to requirements for the quantification of a downturn LGD for advanced IRB under Basel II, we analyzed the implications of our model for this purpose, finding the later to be *declining* for higher expected LGD, higher for lower ranked instruments, and increasing in the correlation between the process driving firm default and recovery on collateral. Finally, we validated our leading model derived herein in an out-of-sample bootstrapping exercise, comparing it to two alternatives, a high-dimensional regression model and a non-parametric benchmark, both based upon the same URD data. We found our model to compare favorably in this exercise.

We conclude that our model is worthy of consideration to risk managers, as well as supervisors concerned with advanced IRB under the Basel II capital accord. It can be a valuable benchmark for internally developed models for ultimate LGD, as this model can be calibrated to LGD observed at default (either market prices or model forecasts, if defaulted instruments non-marketable) and to ultimate LGD measured from workout recoveries. Risk managers can use our model as an input into internal credit capital models.

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